Present Bias in Consumption-Saving Models: A Tractable Continuous-Time Approach*

PRELIMINARY DRAFT

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Abstract

This paper studies the consumption-saving decisions of present-biased consumers. Building on Harris and Laibson (2013), I show that continuous-time methods allow for present bias to be tractably incorporated into incomplete markets models. First, I solve a workhorse Aiyagari-Bewley-Huggett model with present-biased consumers. The equilibrium with present bias features a larger mass of low-liquidity households and a higher aggregate marginal propensity to consume (MPC), but also a thicker right tail of high-wealth households. Second, I extend the model to include credit cards, illiquid assets, and naivete. In this rich economic environment I present closed-form expressions characterizing the effect of present bias on consumption, the demand for illiquid assets, and welfare. This welfare analysis specifies the channels through which present bias can matter for policy, and leads to what I call the present-bias dilemma: present bias has large welfare costs, but individuals have little ability to alleviate these costs without government intervention.

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1   Introduction

There is widespread evidence that consumers exhibit "present bias" across a variety of decision-making contexts. This evidence exists both in the lab (Ainslie, 1992; Frederick et al., 2002; Cohen et al., 2020), and in field settings ranging from credit card usage (Meier and Sprenger, 2010) to retirement savings decisions (Madrian and Shea, 2001) to labor effort (DellaVigna and Paserman, 2005; Augenblick et al., 2015).

Despite the empirical evidence that consumers exhibit present bias, the modeling of consumption-saving behavior has been slow to incorporate these insights. This is because the literature is stuck at an impasse. On the one hand, in stylized "pencil-and-paper" models in which the consumption function is linear in wealth (e.g., no income uncertainty and no borrowing constraints), the consumption of present-biased agents is observationally equivalent to the consumption of exponential agents (Laibson, 1996; Barro, 1999). On the other hand, though present bias can produce a variety of behaviors in richer economic environments that must be solved numerically, the equilibrium to these models is often difficult to characterize in practice.[1] Even when well-behaved numerical solutions do exist, there is often little theoretical justification for the resulting economic dynamics.

The contribution of this paper is to show that continuous-time methods are a path forward. By recasting present bias in continuous time, this paper develops both the analytical and the numerical methods to tractably solve incomplete markets models with present-biased consumers. I use these methods to present a set of new theoretical results characterizing the consumption-saving behavior of present-biased agents.

The paper proceeds in two steps. First, I study present bias in the workhorse "Aiyagari-Bewley-Huggett" incomplete markets model. The goal of this step is to build intuition for present bias in the standard heterogeneous-agent model that economists are familiar with, and to highlight the tractability gained by recasting present bias in continuous time. Second, I extend the model to include features such as credit card debt, illiquid assets, and naivete.

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[1]Present bias can generate strategic interactions between selves, making consumption-saving decisions the equilibrium outcome of a dynamic intrapersonal game. These sorts of models often produce equilibrium non-uniqueness and consumption pathologies (i.e., highly sensitive consumption functions that feature non-monotonicities and downward discontinuities). Both of these issues have made models with present bias difficult to solve in practice (Harris and Laibson, 2001, 2003; Krusell and Smith, 2003; Chatterjee and Eyigungor, 2016; Cao and Werning, 2018; Laibson and Maxted, 2020).

In this extended model, continuous-time methods allow me to derive closed-form solutions characterizing the effect of present bias on consumption-saving decisions, the effect of present bias on the demand for illiquid assets, and the welfare cost of present bias.

In discrete time, present-biased preferences are characterized by the quasi-hyperbolic discount function: $1, \beta\delta, \beta\delta^2, \beta\delta^3, \dots$. Short-run discount factor β creates a disproportionate focus on the present period by driving a wedge between utility earned "now" and utility earned "later." Whenever $\beta < 1$, preferences are time inconsistent. In the context of consumption-saving models, present bias implies that each self overconsumes relative to the preferences of any other self.

The modeling of present bias also requires an assumption about the extent to which agents are aware of their self-control problems (O'Donoghue and Rabin, 1999, 2001). "Sophisticated" agents are fully aware of their dynamic inconsistency. "Partially naive" agents underestimate the magnitude of their self-control problems, and instead expect (incorrectly) that all future selves will discount according to the discount function: $1, \beta^E\delta, \beta^E\delta^2, \beta^E\delta^3, \dots$, where $\beta^E \in (\beta, 1)$. In the limiting case of $\beta^E = 1$, "fully naive" agents perceive that all future selves will behave in a perfectly time-consistent manner. For all but full naivete, present-biased agents disagree with the expected behavior of future selves. This dynamic disagreement means that the behavior of present-biased agents is the equilibrium outcome of a dynamic game played by different temporal selves of the consumer (Strotz, 1956; Laibson, 1997). For methodological purposes I begin by assuming that agents are "sophisticated" ($\beta^E = \beta$), but the model will later be generalized to allow for naivete.

Building on Harris and Laibson (2013), this paper studies present bias in the continuous-time model that results when the length of each period is taken to zero. The continuous-time specification of present bias is referred to as *Instantaneous Gratification* (IG), because each self lives for a vanishingly short period of time and discounts all future selves discretely by β. While the assumption that each self lives for a single instant is made for mathematical convenience, Laibson and Maxted (2020) show that IG preferences closely approximate discrete-time models with period lengths that are psychologically appropriate.[2]

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[2] As detailed in Section 2, laboratory studies find that the temporal division between "now" and "later" is less than one week. However, discrete-time consumption-saving models typically use either quarterly or annual time-steps that are inconsistent with the high frequency at which present bias operates.

The first part of this paper presents a benchmark incomplete markets model similar to Bewley (1986), Huggett (1993), and Aiyagari (1994) (see Section 3).[3] I follow Achdou et al. (2020) and set the model in continuous time. Incomplete markets models have proven difficult to solve under present bias because the consumption decisions of present-biased agents are themselves the equilibrium outcome of a dynamic intrapersonal game. The general equilibrium to these models therefore takes the form of a nested equilibrium. The "inner equilibrium" takes market prices as given and solves for the equilibrium to the consumer's intrapersonal game. The "outer equilibrium" finds the prices that clear markets when the individual-level policy functions are aggregated. Extending an important insight of Achdou et al. (2020), the benefit of continuous time is that the nested equilibrium can be characterized by a system of two coupled partial differential equations.

I numerically solve the workhorse Aiyagari-Bewley-Huggett model for exponential preferences ($\beta = 1$) and for sophisticated present-biased preferences ($\beta < 1$). Both models are calibrated to match the same average wealth level. Despite average wealth being held constant, wealth inequality is strongly amplified in the present-biased calibration: poor agents save less, while rich agents save more. The decreased saving of poor agents produces a larger share of low-liquidity agents who struggle to maintain buffer stocks, and a higher aggregate MPC. The increased saving of wealthy agents produces a wealth distribution with a thicker right tail.

Relative to the $\beta = 1$ discount function, the calibrated $\beta < 1$ discount function features less patience in the short run, but more patience in the long run. Which effect dominates depends on whether the agent is poor or rich. To formalize this result, I present a generalized Euler equation for IG agents. This equation shows that IG agents have an effective discount rate that varies with their MPC. Intuitively, high MPCs discourage saving because a marginal dollar of savings will be quickly (over)consumed by future selves.[4] When markets are incomplete, the consumption function is steeper for low-liquidity agents than for high-liquidity agents. Accordingly, present-biased consumers will act relatively more impatiently near the borrowing constraint, and relatively more patiently as they accumulate wealth.

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[3]Seminal work also includes Imrohoroğlu (1989), Zeldes (1989), Deaton (1991), Carroll (1997), and Gourinchas and Parker (2002).

[4]This result is similar to the discrete-time Hyperbolic Euler Relation of Harris and Laibson (2001).

The second part of this paper is conducted in partial equilibrium (see Sections 5 and 6). I leverage the tractability of IG preferences to develop a new set of theoretical results characterizing the behavior of present-biased consumers. These results rely on two key assumptions: (i) individuals have constant relative risk aversion (CRRA) utility; and (ii) the borrowing limit never binds in equilibrium (e.g., it is the natural borrowing limit). When these assumptions hold, I show that the IG agent's intrapersonal equilibrium can be characterized directly from the behavior of a standard exponential agent. This simple but powerful observation allows for the *relative* effect of present bias to be expressed in closed form, even in complex models that must be solved numerically.

Though the assumption of non-binding borrowing limits may seem strong, I first extend the workhorse model so that it allows for arbitrarily onerous interest rates on borrowing (i.e., "soft borrowing constraints"). That is, the model allows for the interest rate on borrowing to be larger than the interest rate on saving, and for the interest rate on borrowing to become arbitrarily large as the agent borrows more. I show in this environment that $\beta < 1$ provides a strong motive for consumers to accumulate high-interest debt.

Next, I extend the model to include an illiquid asset in addition to the liquid asset. Asset illiquidity is an important feature of state of the art heterogeneous-agent models, such as the HANK model of Kaplan et al. (2018), in order to generate "wealthy hand-to-mouth" households. Asset illiquidity is also a central topic for research on present bias, which argues that present-biased consumers seek out illiquid wealth as a commitment device to increase saving and limit overconsumption (Strotz, 1956; Laibson, 1997; Angeletos et al., 2001; Amador et al., 2006). This research has been influential in the policy sphere, encouraging the use of illiquid accounts to increase retirement and rainy-day savings (see, for example, the discussion of "life-cycle myopia" in Feldstein and Liebman, 2002).

In contrast to this research, I show in a general incomplete markets model that present bias does not necessarily affect the demand for illiquid assets. Provided that the borrowing constraint does not bind in equilibrium, present-biased consumers do not seek out illiquidity because illiquid assets do not actually limit overconsumption. Intuitively, attempting to reduce overconsumption by imposing asset illiquidity is like playing a game of Whack-a-Mole. Since this is a two-asset model, the *illiquid* asset is never needed to fund current

consumption. Instead, the agent can always increase their consumption by adjusting their holdings of the *liquid* asset. Indeed, the existence of a liquid asset completely undoes any commitment properties of the illiquid asset. Retirement systems around the world rely on illiquidity to incentivize retirement savings (Beshears et al., 2015). The results in this paper cast doubt on the benefits of such policies.

Third, I extend the model again to allow the consumer to be naive about their self-control problems. The consumption choices of present-biased agents have only been characterized in simple economic environments that can be solved by hand. An open question is how present bias affects consumption-saving decisions in more realistic environments. In this general model featuring stochastic income, flexible soft borrowing constraints, liquid and illiquid assets, and potential naivete, the effect of present bias on consumption can be characterized with a simple closed-form expression. Let β denote the agent's true short-run discount factor, let $\beta^E \in [\beta, 1]$ denote their perceived present bias, and let γ denote the coefficient of relative risk aversion. If a standard exponential agent consumes $\check{c}$, a present-biased agent will consume $\left(\frac{\beta^E}{\beta}\right)^{\frac{1}{\gamma}} \left(\frac{\gamma}{\gamma-(1-\beta^E)}\right) \times \check{c}$.

This simple consumption equation can also be used to characterize the effect of naivete on consumption-saving decisions. If a sophisticated agent consumes c, a naive agent will consume $\left(\frac{\beta^E}{\beta}\right)^{\frac{1}{\gamma}} \frac{\gamma-(1-\beta)}{\gamma-(1-\beta^E)} \times c$. Note that this implies an observational equivalence between sophisticates and naifs. A sophisticated agent with short-run discount factor β will consume identically to a partially naive agent with perceived short-run discount factor β^E and true short-run discount factor $\beta' = \beta^E \left[\frac{\gamma-(1-\beta)}{\gamma-(1-\beta^E)}\right]^{\gamma}$. One takeaway from this observational equivalence is that it will be difficult to identify sophistication versus naivete using data on consumption choices. Equivalently, the assumption of sophistication versus naivete may not be particularly important for the predictions of calibrated consumption-saving models.

Finally, I show that IG preferences allow for a closed-form characterization of the welfare cost of present bias in this extended model with stochastic income, costly borrowing, assets of varying liquidity, and possible naivete. I consider the following experiment in order to present a welfare metric that applies in this general consumption-saving environment. Assume that there exists a perfect commitment device that forces all future selves to behave with complete self-control ($\beta = 1$), but this device costs a perpetual consumption tax of τ. Then, the welfare

cost of present bias is equivalent to a perpetual consumption tax of $\tau = 1 - \left(\frac{\alpha^\gamma}{1-\gamma+\gamma\alpha}\right)^{\frac{1}{1-\gamma}}$, where $\alpha = \left(\frac{\gamma-(1-\beta^E)}{\gamma}\right)\left(\frac{\beta}{\beta^E}\right)^{\frac{1}{\gamma}}$.

This welfare cost is large. Under a relatively conservative calibration with $\beta = \beta^E = 0.75$ and $\gamma = 2$, the cost of present bias is equivalent to a perpetual 2% consumption tax. Under full naivete ($\beta^E = 1$), this cost rises to 2.4%. If $\beta = 0.5$ and $\beta^E = 1$, as estimated in Laibson et al. (2020a), the welfare cost of present bias is equivalent to a perpetual consumption tax of 17.2%. These costs are at least an order of magnitude larger than back-of-the-envelope estimates of the welfare cost of business cycles (Lucas, 1987), and sit at the upper end of calculations in the literature (e.g., Storesletten et al., 2001; Krebs, 2007; Krusell et al., 2009).

From the perspective of an individual consumer, this welfare result leads to what I refer to as the *present-bias dilemma*: though the welfare cost of present bias is large, it is also difficult for an individual to reduce. One corollary of my welfare characterization is that self-imposed financial commitment devices, such as penalty borrowing rates and asset illiquidity, improve the welfare of an IG agent if and only if they also improve the welfare of a $\beta = 1$ agent. This corollary provides an answer to an outstanding empirical puzzle that asks why present-biased agents do not make commitments (e.g., Laibson, 2015): commitment devices may not actually make the present-biased agent better off. Though self-imposed financial penalties do improve incentives, they do not generate perfect commitment. The benefit of improved incentives is always dominated by the added financial cost of the penalized behavior that still does occur.

Crucially, government interventions differ from the sorts of financial commitments that any individual can self-impose. Governments can not only impose corrective taxes (which alone do not improve welfare), but can also redistribute revenues back to consumers. Unlike financial penalties alone, the combination of penalties plus redistribution can be welfare-improving. Since the welfare cost of present bias is both large and difficult for any individual to mitigate, the present-bias dilemma provides an important justification for government interventions that can alleviate present-bias internalities.

Related Literature. The methodological goal of this paper is to show that continuous-time IG preferences are an effective tool for modeling present bias tractably. These methods

are applied and further developed in Laibson, Maxted and Moll (2020b), who use (naive) continuous-time IG preferences to model the consumption response of present-biased households to fiscal and monetary policy. That paper builds a two-asset model with liquid wealth and illiquid home equity, and evaluates the impact of present bias on households' consumption and mortgage-refinancing decisions.

The IG model was first developed in Harris and Laibson (2013). Laibson and Maxted (2020) show that discrete-time models with short period lengths (e.g., 1 week) are closely approximated by IG models. Other papers studying present bias in continuous time include Barro (1999), Luttmer and Mariotti (2003), and Cao and Werning (2016).

One reason that continuous-time IG preferences are useful is that discrete-time models of present bias suffer from consumption pathologies and equilibrium non-uniqueness. These equilibrium features make discrete-time models of present bias difficult to characterize. Harris and Laibson (2001), Harris and Laibson (2003), Cao and Werning (2018), and Laibson and Maxted (2020) discuss these issues in detail.

In the first step, this paper studies an Aiyagari-Bewley-Huggett model in continuous time, building directly on Achdou et al. (2020). Maliar and Maliar (2006) solve a similar model in discrete time but are forced to make smoothness assumptions, which are only valid for β near 1, in order to solve the model. Such assumptions are not needed in continuous time. Angeletos et al. (2001) and Laibson et al. (2020a) also study discrete-time consumption-saving models with present-biased agents.

In the second step, I extend the model to include soft constraints and an illiquid asset, similar to Kaplan et al. (2018). My theoretical results on high-interest borrowing add to a large literature studying how present bias encourages short-term borrowing on unsecured accounts such as credit cards (Ausubel, 1991; Laibson et al., 2003; Heidhues and Kőszegi, 2010; Meier and Sprenger, 2010; Gathergood, 2012; Kuchler and Pagel, 2020) and payday loans (Skiba and Tobacman, 2018; Allcott et al., 2020). I also study the interaction of present bias with asset illiquidity, building on papers such as Strotz (1956), Laibson (1997), Laibson et al. (1998), Amador et al. (2006), Galperti (2015), Bond and Sigurdsson (2018), Moser and Olea de Souza e Silva (2019), and Beshears et al. (2020).[5]

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[5]See also Attanasio et al. (2020) and Kovacs and Moran (2020) for an analysis of temptation preferences.

I then generalize the model to allow for naivete. Seminal work on naivete includes Akerlof (1991) and O'Donoghue and Rabin (1999, 2001). Naivete has been shown to have large effects on contract-choice decisions (DellaVigna and Malmendier, 2004, 2006; Gabaix and Laibson, 2006; Heidhues and Kőszegi, 2010). However, little is known about the effect of naivete on consumption decisions. One exception is Tobacman (2007), who derives an Euler equation for naive consumers under the assumption that the consumption function is differentiable.

Finally, I use IG preferences to characterize the welfare cost of present bias. For a discussion of welfare in models with time-inconsistent preferences, see Bernheim and Rangel (2009) and Bernheim and Taubinsky (2018). This analysis also relates to the more general literature studying present-biased agents' demand for commitment. For overviews, see DellaVigna (2009), Bryan et al. (2010), Laibson (2015), Bernheim and Taubinsky (2018), and Carrera et al. (2020). Corrective taxation of behavioral agents is studied in papers such as Gruber and Köszegi (2001), O'Donoghue and Rabin (2006), Mullainathan et al. (2011), Allcott et al. (2019), Farhi and Gabaix (2020), and Lockwood (Forthcoming).

2 Instantaneous Gratification: A Summary

I begin by summarizing the model of *Instantaneous Gratification* (IG) time preferences. In discrete time, the quasi-hyperbolic discount function is given by: $1, \beta\delta, \beta\delta^2, \beta\delta^3, \dots$. This discount function captures "present bias," because the current self discounts the utility of all future selves by β. IG preferences are the continuous-time limit of this discount function, where each self lives for a vanishingly short length of time. The IG specification of present bias was first presented in Harris and Laibson (2013).

Let the current period be denoted t. Taking the limit of the discrete-time discount function, IG time preferences are given by the following discount function $D(s)$ for $s \geq t$:

$$D(s) = \begin{cases} 1 & \text{if } s = t \\ \beta e^{-\rho(s-t)} & \text{if } s > t \end{cases}. \tag{1}$$

Parameter ρ is the exponential discount rate. Parameter $\beta \leq 1$ is the short-run discount

factor, which drives a wedge between utility earned "now" and utility earned "later." When $\beta < 1$, discount function $D(s)$ features a discontinuity at $s = t$. This is because the current self lives for only a single instant, and discounts all future selves by β. For reference, Figure 1 plots $D(s)$ for two calibrations. The black curve plots a standard exponential discount function with $\beta = 1$ and $\rho = 3.5\%$. The red curve plots an IG discount function with $\beta = 0.75$ and $\rho = 2.5\%$. These are the two time-preference calibrations that I will study numerically in the Aiyagari-Bewley-Huggett model of Section 3.3.
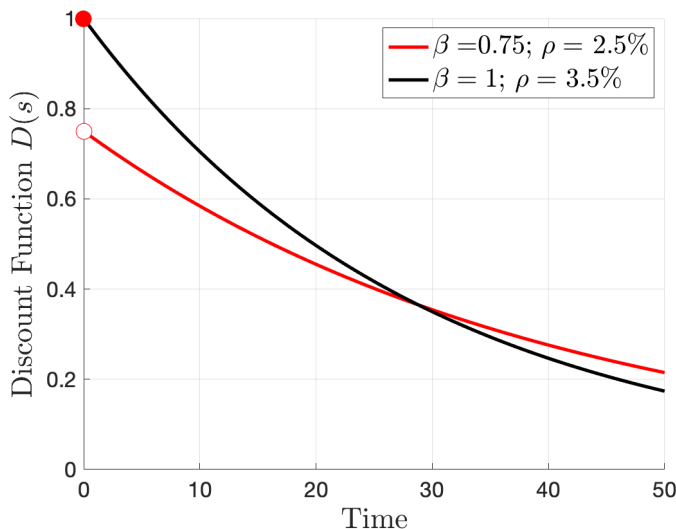

Figure 1: **Discount function** $D(s)$. The red curve plots an IG discount function with $\beta = 0.75$ and $\rho = 2.5\%$. The black curve plots a standard exponential discount function with $\beta = 1$ and $\rho = 3.5\%$. When $\beta < 1$ the IG discount function features a discontinuity between "now" and "later."

As discussed in Laibson and Maxted (2020), IG preferences should be thought of as a mathematically tractable limit case, not as a psychologically realistic model of the discount function. The temporal division between "now" and "later" is certainly longer than a single instant dt. However, this temporal division is also unlikely to extend to the quarterly or annual horizon that discrete-time models typically use. Augenblick (2018) estimates that the division between "now" and "later" is approximately 2 hours. Augenblick and Rabin (2019) find that essentially all discounting occurs within one week. Using fMRI data, McClure et al. (2007) estimate that food rewards are discounted by 50% over a one-hour horizon.[6]

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[6]See also DellaVigna (2018) for a discussion.

Laibson and Maxted (2020) show that IG models provide a close approximation to discrete-time consumption-saving models with time-steps that are psychologically appropriate (i.e., each period lasts for one week or less). This paper takes as given the validity of the IG approximation, and uses IG preferences to tractably incorporate present bias into incomplete markets models.[7]

An important property of the IG discount function is that discounting at short horizons is relatively sensitive to β, while discounting at long horizons is relatively sensitive to ρ. For all $s \geq t$:

$$\frac{\partial \ln(D(s))}{\partial(-\ln(\beta))} = -1, \text{ and}$$
$$\frac{\partial \ln(D(s))}{\partial \rho} = -s.$$

Thus, the elasticity of $D(s)$ with respect to short-run discount rate $-\ln(\beta)$ is constant, while the elasticity of $D(s)$ with respect to long-run discount rate ρ is increasing in horizon s.

Intrapersonal Equilibrium. IG preferences are time inconsistent when $\beta < 1$. As long as the agent is at least partially aware of their self-control problems ($\beta^E < 1$), each self disagrees with the perceived consumption choices of future selves. This means that consumption is modeled as a dynamic intrapersonal game played by different "selves" of the same consumer (Strotz, 1956; Pollak, 1968; Laibson, 1997).

Present bias complicates the general equilibrium analysis of consumption-saving models because each consumer's policy functions are the result of an intrapersonal game for which an equilibrium needs to be derived. Taking prices as given, the equilibrium of this intrapersonal game will be referred to as an *intrapersonal equilibrium*. In general equilibrium, the policy functions that are produced from consumers' intrapersonal equilibria must also clear markets at the aggregate level.

I follow Harris and Laibson (2013) in studying stationary Markov-perfect equilibria to

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[7]This discussion suggests that one could also study present bias in discrete time, but with short (e.g. daily) period lengths. There are two drawbacks to this approach relative to continuous time. First, discrete-time models with short time-steps can be very slow to solve numerically, whereas continuous-time numerical methods are fast (Achdou et al., 2020). Second, the discrete-time specification cannot be easily characterized analytically, while this paper uses IG preferences to prove a new set of theoretical results about present bias.

the intrapersonal game (Maskin and Tirole, 2001).[8] For the models analyzed in this paper, a critical property of IG preferences is that the intrapersonal equilibrium is unique under this refinement. The intrapersonal equilibrium satisfies a partial differential equation, and well-developed numerical methods exist for characterizing this equilibrium (see Appendix A). This is in contrast to discrete time, where equilibria may not be unique and the equilibrium identified by numerical methods often contains pathological properties.[9]

For expositional purposes this paper adopts the baseline assumption that the agent is sophisticated about their present bias ($\beta^E = \beta$). The assumption of sophistication is generalized in Section 5.3 to allow for naivete. Economists have differing views on the extent to which agents are aware of their self-control problems (see e.g. DellaVigna (2018) for a discussion). A prevailing consensus is that partial naivete is most likely, with the degree of naivete being higher in novel environments and lower in recurrent situations (Allcott et al., 2020). I do not take a strong stand on this issue. For all levels of naivete, this paper provides a general method for solving models with present-biased agents that is analytically and numerically tractable, robust to consumption pathologies, and features a unique equilibrium.

3   The Benchmark Aiyagari-Bewley-Huggett Model

The first step of this paper is to embed IG preferences into the workhorse "Aiyagari-Bewley-Huggett" heterogeneous-agent model. I use the continuous-time specification of Achdou et al. (2020). The Aiyagari-Bewley-Huggett model is the standard heterogeneous-agent model that macroeconomists are familiar with. Though stylized, the model serves as an important building-block for a wide range of quantitative applications.[10]

I study an endowment economy in which a continuum of agents have heterogeneous income and wealth profiles. Consumers are able to self-insure against income fluctuations by accumulating a buffer-stock of savings. At the aggregate level there exists an exogenous

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[8]The weaker subgame-perfect refinement introduces a variety of interesting equilibria (Laibson, 1994; Bernheim et al., 2015). Though the restriction to Markov-perfect equilibria is strong, an analysis of other equilibrium refinements is beyond the scope of this paper.

[9]See Harris and Laibson (2001, 2003), Krusell and Smith (2003), and Cao and Werning (2016) for discussions of equilibrium pathologies and non-uniqueness. Laibson and Maxted (2020) provide a summary.

[10]See Heathcote et al. (2009), Guvenen (2011), Quadrini and Ríos-Rull (2015), Krueger et al. (2016), De Nardi and Fella (2017), Benhabib and Bisin (2018), and Kaplan and Violante (2018) for surveys.

supply B of bonds. Interest rate r is determined in general equilibrium to equate the supply of savings with B.

When $\beta < 1$, the solution to this model takes the form of a nested equilibrium. There is a sequence of two equilibria that must be solved jointly: (i) the intrapersonal equilibrium of the IG agent, taking prices as given; and (ii) the general equilibrium, in which the individual-level policy functions are aggregated and markets clear. In continuous time, the joint solution to these equilibria takes the form of two coupled PDEs.

3.1 Consumer Problem (Intrapersonal Equilibrium)

Budgets. Let a_t denote an individual's wealth at time t. a_t evolves as follows:

$$da_t = (y_t + ra_t - c_t)dt. \tag{2}$$

y_t is a stochastic endowment income process and c_t is consumption. y_t follows a two-state Poisson process $y_t \in \{y_1, y_2\}$, with $0 < y_1 < y_2$. The income process jumps from state y_1 to y_2 with intensity λ_1, and from y_2 to y_1 with intensity λ_2.[11]

Wealth is subject to the borrowing limit

$$a_t \geq \underline{a}, \tag{3}$$

where $\underline{a} \in [-\frac{y_1}{r}, 0]$.[12] When $\beta < 1$ the equilibrium is sensitive to whether or not this limit binds in equilibrium. Binding constraints shorten the consumer's effective horizon. As will be detailed below, this interacts with present bias to decrease the savings rate near $\underline{a}$.

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[11]The two-state Poisson process is chosen for simplicity. My results can be extended to more general income processes.

[12]The natural borrowing limit is $\frac{-y_1}{r}$. This is the maximum amount that the agent can borrow such that their income is always sufficient to cover their debt service payments.

Utility and Value. Individuals accrue CRRA utility over consumption:

$$u(c) = \begin{cases} \frac{c^{1-\gamma}-1}{1-\gamma} & \text{if } \gamma \neq 1 \\ \ln(c) & \text{if } \gamma = 1 \end{cases}. \tag{4}$$

Consumers have sophisticated *Instantaneous Gratification* (IG) time preferences. Under IG time preferences, the continuation-value function is standard:

$$v_t = \mathbb{E}_t \left[\int_t^\infty e^{-\rho(s-t)} u(c_s) ds \right]. \tag{5}$$

The current-value function is given by

$$w_t = \beta v_t. \tag{6}$$

The current self discounts the utility of all future selves by β. In continuous time, the current self lives for just a single instant. The utility accrued by the current self therefore has no measurable impact on the overall value function, so $w_t = \beta v_t$.[13]

Throughout this paper, I impose the restriction that $\gamma > 1 - \beta$. This ensures that the consumer's desire to smooth consumption (γ) is greater than their time-inconsistency ($1-\beta$).

Intrapersonal Equilibrium. A stationary Markov-perfect equilibrium to the IG agent's intrapersonal problem is characterized by the following differential equation and optimality condition on $a \in [\underline{a}, \infty)$ and $j \in \{1, 2\}$:[14]

$$\rho v_j(a) = u(c_j(a)) + v_j'(a)(y_j + ra - c_j(a)) + \lambda_j(v_{-j}(a) - v_j(a)), \tag{7}$$

$$u'(c_j(a)) = \begin{cases} \beta v_j'(a) & \text{if } a > \underline{a} \\ \max\{\beta v_j'(\underline{a}), \ u'(y_j + r\underline{a})\} & \text{if } a = \underline{a} \end{cases}. \tag{8}$$

Equation (7) defines the continuation-value function v of the IG agent. It says that the

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[13]Heuristically, w_t is given by $w_t = u(c_t)dt + \beta \mathbb{E}_t \left[\int_{t+dt}^\infty e^{-\rho(s-t)} u(c_s) ds \right]$.

[14]Under certain calibrations v will have a convex kink. Viscosity solutions are therefore used.

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instantaneous change in value due to discounting (ρv) must equal the current utility flow ($u(c)$) plus the expected instantaneous change in the value function ($\mathbb{E}dv/dt$).[15]

Equation (8) defines the IG agent's consumption choice. In continuous time, consumption is unconstrained for all $a > \underline{a}$.[16] Whenever consumption is unconstrained, the IG agent sets the marginal utility of consumption equal to the marginal value of current wealth: $u'(c_j(a)) = w'_j(a) = \beta v'_j(a)$. At the borrowing constraint, the optimality condition is refined to ensure that $c_j(\underline{a}) \leq y_j + r\underline{a}$. If $\beta v'_j(\underline{a}) \geq u'(y_j + r\underline{a})$ the agent will choose to set $c_j(\underline{a}) \leq y_j + r\underline{a}$. Otherwise, consumption is constrained to $y_j + r\underline{a}$.

Equations (7) and (8) look similar to the HJB equation that would arise for a standard exponential agent with $\beta = 1$. As in Achdou et al. (2020), the HJB equation of a $\beta = 1$ agent is given by:

$$\rho v_j(a) = \max_c \ u(c) + v'_j(a)(y_j + ra - c) + \lambda_j(v_{-j}(a) - v_j(a)).$$

Present bias alters the consumption optimality condition (equation (8)). The IG agent sets $u'(c_j(a)) = \beta v'_j(a)$ as opposed to $u'(c_j(a)) = v'_j(a)$. In both cases, the current self sets the marginal utility of consumption equal to the marginal value of current wealth. However, under IG preferences the marginal value of wealth is discounted by β. This is because wealth is consumed by future selves.

An important property of IG preferences is that the value function v is unique. This is in contrast to discrete time, where non-uniqueness is known to exist in some deterministic models (Krusell and Smith, 2003; Cao and Werning, 2016), and uniqueness in stochastic models can only be proven for β close to 1 (Harris and Laibson, 2003).

Proposition 1. *The value function $v_j(a)$ of the IG agent's intrapersonal game is unique.*

Proof. See Appendix C. The proof relies on methods that are presented in Section 4. □

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[15]See Laibson and Maxted (2020) for a derivation of (7) as the limit of a discrete-time model.

[16]For all $a > \underline{a}$, the agent can adopt an arbitrarily high consumption rate without violating the borrowing constraint, so long as this rate of consumption persists for a short enough period of time.

3.2 General Equilibrium

To solve for a general equilibrium to this heterogeneous-agent model, the intrapersonal equilibrium of IG agents is aggregated and bond market clearing is imposed. To close the model as simply as possible, I assume that there is an exogenous supply of safe debt $B \in (\underline{a}, \infty)$ that households can hold (Huggett, 1993). It is well known that the model can be closed in alternate ways (e.g., Aiyagari, 1994). This paper focuses on the demand side of the economy, where present-biased preferences interact with incomplete markets. Simplicity is preferred on the supply side for expositional clarity.

Let $g_j(a,t)$ denote the distribution of wealth and income at time t, such that $\int_{\underline{a}}^{\infty} g_1(a,t)da + \int_{\underline{a}}^{\infty} g_2(a,t)da = 1$. Since this is an endowment economy with exogenous income, the one price that must be pinned down in general equilibrium is the interest rate r. Given r, the consumer's intrapersonal equilibrium is described by equations (7) and (8). The resulting policy functions give rise to a Kolmogorov Forward (KF) equation that characterizes the evolution of the aggregate wealth distribution. In a stationary equilibrium the distribution of wealth is constant:

$$ 0 = -\frac{\partial}{\partial a}[s_j(a)g_j(a)] - \lambda_j g_j(a) + \lambda_{-j}g_{-j}(a), \tag{9} $$

where $s_j(a)$ is the savings policy function $s_j(a) = y_j + ra - c_j(a)$. Equations (7) and (8), plus KF equation (9), define a steady state aggregate savings function:

$$ S(r) = \int_{\underline{a}}^{\infty} a g_1(a)da + \int_{\underline{a}}^{\infty} a g_2(a)da. \tag{10} $$

The bond market clears when when $S(r) = B$.

The action in this model occurs on the demand side of the economy, where consumers are heterogeneous. Present bias adds an additional layer of complexity by making the individual's problem itself a dynamic game. The benefit of the continuous-time approach taken in this paper is that the intrapersonal equilibrium can be characterized by a partial differential equation (equation (7)). Following Achdou et al. (2020), a general equilibrium can then be found by coupling the KF equation in (9) with the IG agent's intrapersonal equilibrium. Us-

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ing this simple approach, this paper is among the first to solve a general equilibrium model where consumers have present bias and policy functions are non-linear.[17]

3.3 Model Solution: Present Bias and Consumption-Saving Decisions in Equilibrium

I now solve this workhorse model numerically to compare the equilibrium of an economy with IG consumers ($\beta < 1$) to an economy with exponential consumers ($\beta = 1$). Afterwards, I provide a set of propositions that formalize equilibrium properties of this workhorse heterogeneous-agent model.

Stylized Calibration. I calibrate the economy to roughly reflect the problem of a typical American household. Average income is normalized to one. I set $y_1 = 0.74$ and $y_2 = 1.26$, with a job switching rate of $\lambda_1 = \lambda_2 = 0.19$. This calibration is a two-state discretization of the income process used in Guerrieri and Lorenzoni (2017).[18]

Borrowing constraint $\underline{a} = -\frac{1}{3}$, which corresponds to the average credit limit reported in the 2016 Survey of Consumer Finances (Laibson et al., 2020b). I set the coefficient of relative risk aversion $\gamma = 2$. The exogenous supply of bonds is calibrated to $B = 3$ in order to capture the ratio of wealth to income in the United States (Kaplan et al., 2018).

I target a steady-state interest rate to 3%.[19] The discount function is calibrated internally to produce this interest rate in equilibrium. In the exponential model with $\beta = 1$, $r = 3\%$ is produced by $\rho = 3.5\%$. In the IG calibration I set $\beta = 0.75$. The calibration of $\beta = 0.75$ is a conservative choice in the lifecycle literature, and the results that follow become more

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[17]One relevant paper in discrete time is Maliar and Maliar (2006). In order to solve the model numerically the authors assume that the consumption function is continuous and differentiable. However, consumption pathologies are known to arise in these sorts of discrete-time models (Harris and Laibson, 2001, 2003; Laibson and Maxted, 2020). By using IG preferences, no such approximations are needed here.

[18]Guerrieri and Lorenzoni (2017) assume that log-income follows an AR(1) at a quarterly frequency: $\log(y_{t+1}) = \rho\left(\log(y_t) - \frac{\sigma^2}{2}\right) + \sigma\varepsilon_{t+1}$. Using the estimates of Floden and Lindé (2001), this process is calibrated with persistence $\rho = 0.967$ and variance $\sigma^2 = 0.017$. I convert this quarterly AR(1) into an Ornstein-Uhlenbeck process, and then discretize the Ornstein-Uhlenbeck process into two states using finite-difference methods. I set the income states to ± 1 standard deviation. With only two income states, the non-linearity created by exponentiating log-income is lost.

[19]A downside of this simple model is that all wealth is risk free. A 3% interest rate is somewhere between the return on risk-free assets and the return on risky assets.

stark as β decreases.[20] Given $\beta = 0.75$, $\rho = 2.5\%$ produces a 3% steady-state interest rate.

To solve this model numerically I build on the finite difference methods presented in Achdou et al. (2020). Barles and Souganidis (1991) prove that a finite difference scheme converges to the unique viscosity solution of an HJB equation whenever three conditions are met: (i) monotonicity, (ii) stability, and (iii) consistency. For $\beta < 1$, a finite difference scheme cannot be applied directly to equations (7) and (8) because the monotonicity property is not satisfied. This paper develops novel numerical methods for reestablishing a convergent finite difference scheme when $\beta < 1$. Details are given in Appendix A.

Consumption and Saving. The top panel of Figure 2 plots the consumption function for the $\beta = 0.75$ calibration and the $\beta = 1$ calibration. The $\beta = 1$ consumption function is standard (see Achdou et al. (2020) for details). For $\beta = 0.75$, consumption is well-behaved in that it does not suffer from pathologies on the interior of the wealth space (Laibson and Maxted, 2020). However, the consumption function features a discontinuity at $a = \underline{a}$ when $y = y_1$.[21] This consumption discontinuity is produced by the corresponding discontinuity in discount function $D(s)$. Consider the self in control an instant before the constraint binds. This self does not want to smooth consumption with the next self (for whom the constraint will bind), since the self in control discounts the utility of the next self by β.

The bottom panel of Figure 2 plots the corresponding savings function $s_j(a) = y_j + ra - c_j(a)$ for both calibrations. Near $\underline{a}$ the IG agent has a lower savings rate than the exponential agent. This pattern reverses as a increases. In short, the IG agent saves less when poor but saves more when wealthy. Relative to the $\beta = 1$ agent, the IG agent has both an incentive for lower saving ($\beta < 1$) and an incentive for higher saving (a lower value of ρ). When the consumption function is non-linear, the relative impact of β versus ρ varies over the state space. Near $\underline{a}$, $\beta < 1$ dominates and the IG agent saves less than the exponential agent. Since the relative impact of β versus ρ must exactly offset in general equilibrium (so that $S(r) = B$), away from $\underline{a}$ the low value of ρ dominates and the IG agent saves more than the

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[20]For example, Angeletos et al. (2001) set $\beta = 0.7$, arguing that this is consistent with laboratory experiments. Laibson et al. (2020a) estimate $\beta = 0.5$ in a structural lifecycle model. Allcott et al. (2020) estimate $\beta = 0.75$ on a sample of payday loan users.

[21]These sorts of predictable consumption discontinuities are observed empirically, and are a challenge for rational models to match (e.g., Mastrobuoni and Weinberg, 2009; Ganong and Noel, 2019).
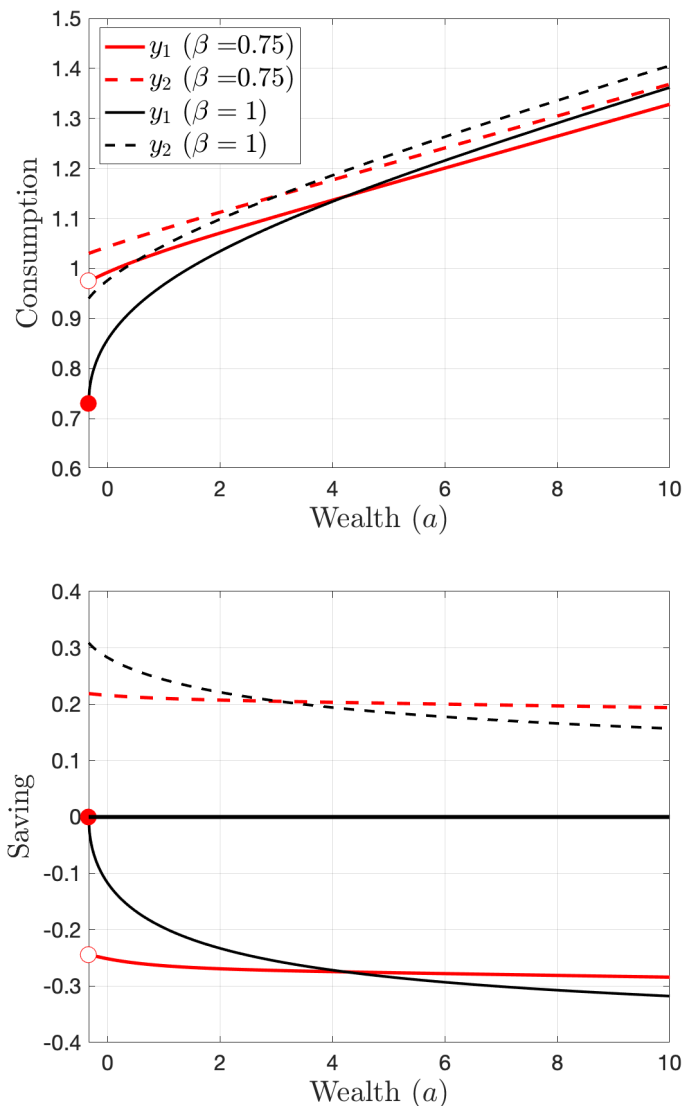
exponential agent.

Figure 2: **Equilibrium Consumption-Saving Decisions.** This figure plots the consumption function (top) and the savings function (bottom) for the $\beta = 0.75$ calibration and the $\beta = 1$ calibration.

There are two reasons why the relative effect of $\beta < 1$ matters more for low levels of wealth. First, present bias creates a disagreement between successive selves that is increasing in the slope of the consumption function (i.e., the instantaneous MPC). Since the consumption function is concave, the slope of the consumption function is highest near $\underline{a}$. This decreases the savings rate near $\underline{a}$. Second, in the case where the borrowing constraint binds, time inconsistency interacts with the consumer's effective planning horizon to lower

the savings rate near $\underline{a}$.[22,23]

The intuition for both effects is as follows. Sophisticated present bias means that the current self distrusts the consumption decisions of future selves. This has offsetting effects on the current self's incentive to save. On the one hand, the current self knows that wealth will be spent imprudently in the future. This decreases the savings rate of self t. On the other hand, because self t knows that future selves will not save enough, self t has an incentive to set aside wealth today in order to buffer against future overconsumption. This increases the savings rate of self t. The relative strength of the second effect depends on both the MPC of future selves and the effective planning horizon. If future selves adopt a high MPC then marginal savings by self t will be quickly consumed, reducing self t's ability to compensate for the overconsumption of future selves. Similarly, binding borrowing constraints shorten the consumer's effective planning horizon by limiting self t's ability to pass wealth far into the future (because any marginal savings will be fully consumed in finite time). This again reduces the incentive to save.

This MPC-dependent savings incentive is formalized in Corollary 3 below, which presents a generalized Euler equation for IG agents. This is the continuous-time analogue to the Hyperbolic Euler Relation presented in Harris and Laibson (2001). Proposition 4 characterizes the interaction between present bias and the planning horizon near $\underline{a}$, which is responsible for the consumption discontinuity at $\underline{a}$.

Distributions. Figure 3 plots the stationary distribution of consumers in the two calibrations. By bond market clearing (with $B = 3$), the aggregate wealth level is constant across the two calibrations. However, the underlying distribution of wealth differs considerably. Relative to $\beta = 1$, the $\beta = 0.75$ calibration features both a larger share of agents near $\underline{a}$ and a thicker right tail. This is consistent with the savings functions shown in Figure 2. Near $\underline{a}$ the IG agent has difficulty generating precautionary savings. However, the lower long-run

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[22]The effective planning horizon can be defined as the minimum length of time over which an additional ϵ of wealth given at time t will alter the equilibrium path of wealth. The effective planning horizon becomes arbitrarily short as the agent nears $\underline{a}$, because any marginal saving by self t only affects the equilibrium path of wealth until the constraint $\underline{a}$ is hit. Alternatively, the effective planning horizon becomes arbitrarily long as $a \to \infty$ (or by calibrating $\underline{a} \leq \frac{-y_1}{r}$).

[23]Note that the borrowing constraint will not necessarily bind in equilibrium (e.g., $\underline{a} = \frac{-y_1}{r}$). When the constraint does not bind, this second effect disappears.

discount rate of the IG agent means that the IG agent also adopts a higher savings rate as a increases. This generates the thicker right tail observed in Figure 3.
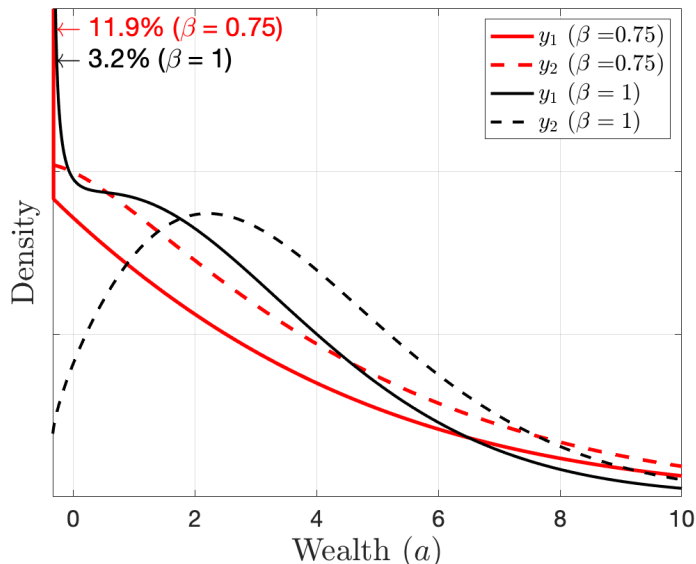

Figure 3: **The Distribution of Wealth.** This figure shows the stationary wealth distribution for the $\beta = 0.75$ calibration and the $\beta = 1$ calibration.

Quantifying these differences, Table 1 compares various measures of wealth inequality across the two calibrations. Wealth inequality is much higher for the $\beta = 0.75$ economy. The maximum wealth level attained in the $\beta = 0.75$ economy is twice as large as the maximum wealth level attained in the $\beta = 1$ economy.[24] The $\beta = 0.75$ economy also features more wealth in the top 0.1%, 1%, 5%, and 10%. However, the $\beta = 0.75$ economy produces almost four times as many agents constrained at $\underline{a}$.[25] As shown in Panel C of the table, all of these differences in wealth inequality arise even under the restriction that aggregate wealth is constant across the two calibrations.

One way to contextualize the results in Table 1 is to note that present bias generates similar effects on the wealth distribution as heterogeneous time preferences.[26] While mod-

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[24]For further intuition, Remark 7 presents a closed-form approximation for the maximum wealth level. It is a known feature of these models that they feature a bounded wealth distribution (Achdou et al., 2020).

[25]Gross and Souleles (2002) estimate that changes to credit limits increase borrowing by 10-14%. Among new credit card applicants, Agarwal et al. (2018) document that credit limit increases generate additional borrowing for the majority of households. Both of these findings are consistent with the existence of a mass of agents who are close to liquidity constraints.

[26]Heterogenous time preferences are a tool that has frequently been employed in macroeconomic models to generate realistic wealth distributions (e.g., Campbell and Mankiw, 1989; Krusell and Smith, 1998).

	$\beta = 1$	$\beta = 0.75$
Panel A: Upper Wealth Moments		
Maximum Wealth (a_{max})	71.2	154.4
Average Wealth | Top 0.1 %	14.7	21.9
Average Wealth | Top 1 %	11.7	16.7
Average Wealth | Top 5 %	9.3	12.5
Average Wealth | Top 10 %	8.1	10.6
Panel B: Lower Wealth Moments		
Share $a < 0$	8.0%	18.0%
Share $a = \underline{a}$	3.2%	11.9%
Average Wealth | Bottom 50 %	1.1	0.5
Panel C: Aggregate Wealth Moments		
Standard Deviation a	3.8	4.5
Mean a	3.0	3.0
Median a	2.6	2.0

Table 1: **Wealth Distribution Moments.** This table characterizes the wealth distribution for the $\beta = 0.75$ calibration and the $\beta = 1$ calibration.

els with heterogeneous time preferences impose preference heterogeneity *across* individuals, present bias instead generates effective time-preference heterogeneity *within* individuals that varies with wealth. Near $\underline{a}$ the IG agent acts impatiently. This produces a large mass of constrained agents. IG preferences simultaneously generate a long right tail in the wealth distribution, since the IG agent acts more patiently as their wealth increases. It is well-known that heterogeneous-agent models of the type solved here produce counterfactually low levels of wealth inequality when $\beta = 1$ (Carroll, 1997; Quadrini et al., 1997; Quadrini, 1999). While this numerical example is far too stylized to make any quantitative claims, present bias moves these models in the right direction.[27]

The Marginal Propensity to Consume (MPC). Uninsurable income shocks generate state-dependent MPCs. To study consumers' MPCs in this model, I follow Achdou et al. (2020) who define the MPC as follows:

Definition 2 (Achdou et al. (2020) Definition 1)**.** *The Marginal Propensity to Consume*

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[27]Mian et al. (2020) document that wealthy households have higher savings rates than poor households, and Fagereng et al. (2019) find that (net) savings rates are approximately constant in wealth. Though the IG model cannot replicate these facts, the IG model performs better than the exponential model because the savings rate declines more slowly in wealth for the IG calibration.

over a period τ is given by

$$MPC_{j,\tau}(a) = C'_{j,\tau}(a), \ where \tag{11}$$

$$C_{j,\tau}(a) = \mathbb{E}_t\left[\int_t^{t+\tau} c_j(a_s)ds \mid a_t = a, y_t = y_j\right]. \tag{12}$$

In equation (11) the MPC is defined over a discrete unit of time τ. While one could also study the instantaneous MPC of $c'_j(a)$, the cumulative MPC is more empirically relevant because consumption is typically observed at a quarterly and/or annual horizon. The MPC can be computed numerically using the Feynman-Kac formula.

Figure 4 plots the quarterly MPC in the two calibrations. Near $\underline{a}$ the MPC is larger for the IG agent because the IG agent lacks the self-control to smooth consumption into the borrowing limit. Away from $\underline{a}$ the MPC is smaller for the IG agent. Here the lower value of ρ dominates, and the IG agent adopts a higher savings rate than the exponential agent.
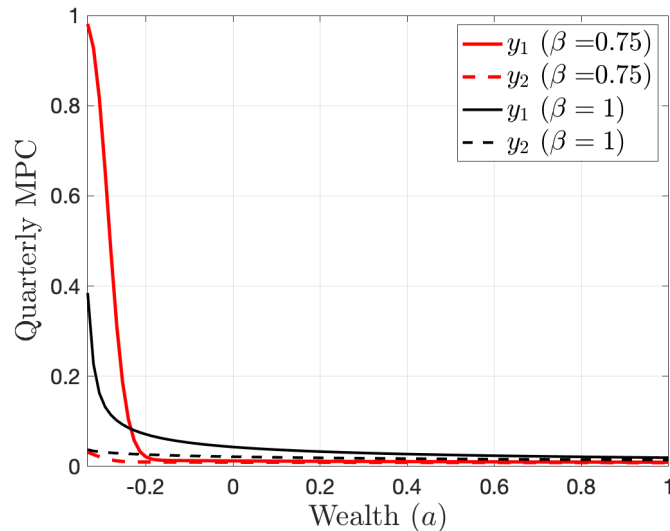
Figure 4: **MPCs.** This figure plots quarterly MPCs for the $\beta = 0.75$ calibration and the $\beta = 1$ calibration.

The MPC analysis in Figure 4 does not account for the distribution of agents across the high- and low-MPC parts of the state space. In particular, there is more mass near $\underline{a}$ in the $\beta = 0.75$ calibration than in the $\beta = 1$ calibration. In the $\beta = 0.75$ calibration, the average MPC is 24.6% for low-income agents, and 0.9% for high-income agents. In the

$\beta = 1$ calibration, the average MPC is 4.5% for low-income agents, and 1.2% for high-income agents. Though this example is stylized, the stark difference in MPCs for low- versus high-income agents when $\beta = 0.75$ suggests that fiscal stimulus targeted to low-income households will be particularly effective when consumers are present-biased. A more complete analysis of the effects of present bias on fiscal policy is presented in Laibson et al. (2020b).

3.4 Theoretical Properties

I now characterize consumption-saving properties of the model's equilibrium. While the results in this section are specific to the economic environment modeled above, they can typically be generalized. Unless given in the main text, all proofs are provided in Appendix C. Note that these proofs rely on methods that are not presented until Section 4.

Wealth-Dependent Discounting: An Euler Equation. Equipped with Proposition 1, the following corollary characterizes the continuous-time Euler equation for the IG consumer.

Corollary 3. *Whenever $c_j(a)$ is locally differentiable in a, consumption satisfies the following Euler equation:*

$$\frac{\mathbb{E}_t \left(du'(c_j(a_t))/dt \right)}{u'(c_j(a_t))} = \left[\rho + (1 - \beta)c'_j(a_t) \right] - r. \tag{13}$$

The lefthand side of equation (13) is the growth rate of marginal utility. When $\beta = 1$, the growth rate of marginal utility equals $\rho - r$. When $\beta < 1$, dynamic inconsistency means that the IG agent also cares about the extent to which the next self will consume out of a marginal dollar of savings, as captured by the instantaneous MPC of $c'_j(a_t)$. This dynamic disagreement creates a time-varying effective discount rate of $\rho + (1 - \beta)c'_j(a_t)$.

Corollary 3 highlights why present bias is not observationally equivalent to exponential discounting when the consumption function is non-linear. The term $(1-\beta)c'_j(a_t)$ is a measure of the consumer's relative time-inconsistency. If consumption is linear then $c'_j(a_t)$ is constant. If consumption is non-linear then the consumer's self-control varies over the state space. In particular, present-biased consumers will act more impatiently near the borrowing constraint,

because the high MPC of future selves lowers the current self's willingness to save.[28]

Corollary 3 is the continuous-time equivalent of the discrete-time Hyperbolic Euler Relation derived in Harris and Laibson (2001). The benefit of continuous time is tractability. In discrete time, consumption pathologies force Harris and Laibson (2001) to use bounded-variation calculus to characterize the Hyperbolic Euler Relation. Continuous-time methods provide a much simpler equation.

Consumption Behavior of the Poor. Present bias interacts with the consumer's effective horizon. This makes consumption sensitive to whether or not the borrowing constraint binds in equilibrium. In particular, a binding constraint will produce a consumption discontinuity at $\underline{a}$, as is formalized in the following proposition.

Proposition 4. *Let $c_j(\underline{a}+) = \lim_{a \to^+ \underline{a}} c_j(a)$. If $\beta < 1$ and $\underline{a}$ binds for income state j then there is a discontinuity in consumption at $\underline{a}$, such that $c_j(\underline{a}+) > c_j(\underline{a})$. Specifically, $c_j(\underline{a}) = y_j + r\underline{a}$ while $c_j(\underline{a}+)$ is defined implicitly by*

$$u'(c_j(\underline{a}+)) = \beta \frac{u(c_j(\underline{a}+)) - u(y_j + r\underline{a})}{c_j(\underline{a}+) - (y_j + r\underline{a})}. \tag{14}$$

Proposition 4 formalizes the discontinuity in $c_1(a)$ seen in Figure 2. The IG agent does not smooth consumption into the constraint. Instead, the IG agent chooses a high level of consumption until the instant that the constraint binds, at which point consumption drops discretely. The following perturbation argument, first provided in Harris and Laibson (2004), provides the intuition for equation (14). Consider the self who lives one instant before the borrowing constraint binds. That self can cut consumption by da at a utility cost of $u'(c_j(\underline{a}+))da$. This allows future selves to consume at rate $c_j(\underline{a}+)$ rather than $y_j + r\underline{a}$ for a timespan of $dt = \frac{da}{c_j(\underline{a}+)-(y_j+r\underline{a})}$. The current self values this additional consumption at $\beta \left(u(c_j(\underline{a}+)) - u(y_j + r\underline{a}) \right) dt$. The current self must be indifferent to this perturbation in equilibrium, implying $u'(c_j(\underline{a}+))da = \beta \left(u(c_j(\underline{a}+)) - u(y_j + r\underline{a}) \right) dt$. Plugging in for dt yields equation (14).

---

[28]This result has some conceptual similarities to the temptation model of Banerjee and Mullainathan (2010) in that differences in patience are the endogenous outcome of differences in wealth (contrary to the typical intuition that differences in wealth are the endogenous outcome of differences in patience).

Consumption Behavior of the Wealthy. The sole source of uncertainty in this model is income risk. Because income risk does not scale with wealth, income risk ceases to affect consumption-saving decisions in the limit as $a \to \infty$. Under exponential discounting, consumption and savings are asymptotically linear in a (Achdou et al., 2020). This linearity property also applies to IG agents.[29]

Proposition 5. *Consumption and saving are asymptotically linear in a. Specifically,*

$$\lim_{a \to \infty} c_j(a) = \frac{\rho - (1 - \gamma)r}{\gamma - (1 - \beta)}a \tag{15}$$

$$\lim_{a \to \infty} s_j(a) = \frac{r\beta - \rho}{\gamma - (1 - \beta)}a \tag{16}$$

An interesting corollary is that when $\beta < 1$ the elasticity of intertemporal substitution (EIS) is no longer given by $\frac{1}{\gamma}$.

Corollary 6. *In the limit as $a \to \infty$, the EIS is given by:*

$$EIS = \lim_{a \to \infty} \frac{d\left[\mathbb{E}_t \frac{dc_t/dt}{c_t} \right]}{dr} = \frac{\beta}{\gamma - (1 - \beta)}.$$

A discrete-time version of this result is given in Laibson (1998). With IG preferences, the EIS is less than $\frac{1}{\gamma}$ when $\gamma > 1$, and the EIS is greater than $\frac{1}{\gamma}$ when $\gamma < 1$. The intuition for this result is similar to the Euler equation in Corollary 3. The Euler equation shows that the IG agent chooses their consumption profile for strategic reasons in addition to standard consumption-smoothing considerations. The EIS determines the sensitivity of the IG agent's current consumption to these strategic motives. When $\gamma < 1$ the IG agent responds more than the standard exponential agent to interest rate changes. The reverse is true for $\gamma > 1$.

Proposition 5 also allows for an approximation of the maximum level of wealth that can be attained in this model (see Table 1). Note that this result is an approximation because, for finite wealth, the extent to which consumption can be approximated by a linear function is calibration-dependent.

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[29]These asymptotic consumption functions highlight that the consumption of present-biased agents is observationally equivalent to the consumption of exponential agents when the consumption function is linear in wealth (Laibson, 1996; Barro, 1999). As the numerical results in Section 3.3 clearly illustrate, however, this observational equivalence does not hold over the entire state space when income is stochastic.

Remark 7. *The maximum level of wealth can be approximated as follows:*

$$a_{max} \approx \frac{\kappa(\bar{y}/r) - y_2}{r - \kappa},$$

where $\kappa = \frac{\rho - (1-\gamma)r}{\gamma - (1-\beta)}$ *is the consumption rate in equation* (15) *and* $\bar{y} = \frac{\lambda_2 y_1 + \lambda_1 y_2}{\lambda_1 + \lambda_2}$ *denotes average income.*

The intuition for Remark 7 is straightforward. Using equation (15), a wealthy, high-income, agent will save approximately $s_2(a) = y_2 + ra - \kappa \left(a + \frac{\bar{y}}{r}\right)$. Setting $s_2(a) = 0$ and rearranging yields the desired result.

4 Tractability in Continuous Time: The $\hat{u}$ Agent

The reason that IG preferences are tractable is that the problem of the dynamically inconsistent IG agent can be recast as a dynamically consistent optimization problem. Specifically, the value function of the IG agent is equivalent to the value function of an agent who discounts exponentially ($\beta = 1$), but has a modified utility function denoted $\hat{u}$. This result was first presented in Harris and Laibson (2013), and it is used extensively in this paper.

The $\hat{u}$ construction is critical because the intrapersonal equilibrium of the IG agent can be characterized analytically from the $\hat{u}$ agent. Thus, one does not need to solve directly for the equilibrium of the IG agent. Instead, one can solve for the policy and value functions of the dynamically consistent $\hat{u}$ agent. The IG agent's intrapersonal equilibrium can then be recovered directly from the $\hat{u}$ agent.

Developing this insight one step further, most of the theoretical results presented in Section 5 exploit the following observation: if the borrowing limit does not bind in equilibrium then the $\hat{u}$ utility function is a positive affine transformation of CRRA utility. Accordingly, if the borrowing limit does not bind in equilibrium then the $\hat{u}$ agent's policy functions are identical to those of a standard exponential agent with CRRA utility. When this is the case, the IG agent's equilibrium can be recovered from that of a standard exponential agent. This property is critical for my results. It means that the behavior of the IG agent can be analytically characterized *relative* to an equivalent agent with $\beta = 1$, even in complex models

that must be solved numerically.

The remainder of this section formalizes this discussion.

Constructing Two Additional Agents. I now introduce two new types of agents. These two new agents will aid in characterizing the equilibrium of the IG agent.

Definition 8 ($\hat{u}$ Agent)**.** *The first agent is referred to as a "$\hat{u}$ agent." The $\hat{u}$ agent discounts exponentially ($\beta = 1$) and has a modified utility function, denoted $\hat{u}$ (defined below). The value and policy functions of the $\hat{u}$ agent will be denoted with a hat (e.g., $\hat{v}_j(a)$ and $\hat{c}_j(a)$).*

The $\hat{u}$ agent is reverse-engineered so that the continuation-value function of the $\hat{u}$ agent is equivalent to the continuation-value function of the IG agent. The purpose of the $\hat{u}$ agent is to allow the IG problem to be recast as a dynamically consistent optimization problem.

Definition 9 (Standard Exponential Agent)**.** *The second agent is referred to as a "standard exponential agent." The standard exponential agent discounts exponentially ($\beta = 1$) and has standard CRRA utility $u(c)$. The value and policy functions of the standard exponential agent will be denoted with an upside-down hat (e.g., $\check{v}_j(a)$ and $\check{c}_j(a)$)*

The standard exponential agent has the utility function and time preferences that economists typically work with. The standard exponential agent will provide a useful point of comparison for understanding the non-standard IG and $\hat{u}$ agents.

The $\hat{u}$ Construction. Define

$$\psi = \frac{\gamma - (1 - \beta)}{\gamma}. \tag{17}$$

Note that $\psi \in (0, 1]$.[30] Next, define

$$\hat{u}^{+}(\hat{c}) = \frac{\psi}{\beta} u\left(\frac{1}{\psi}\hat{c}\right) + \frac{\psi - 1}{\beta}. \tag{18}$$

$\hat{u}^{+}(\hat{c})$ is a positive affine transformation of utility function $u(c)$, and is constructed so that $\hat{u}^{+}(\hat{c}) < u(\hat{c})$ for all $\hat{c} > 0$.

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[30]This follows from the parameter restrictions of $\beta \in (0, 1]$ and $\gamma > 1 - \beta$.

The complete $\hat{u}$ utility function depends on whether or not the borrowing constraint binds. $\hat{u}$ is defined as follows:

$$\hat{u}_j(\hat{c}, a) = \begin{cases} \hat{u}^+(\hat{c}) & \text{if } a > \underline{a} \\ \hat{u}^+(\hat{c}) & \text{if } a = \underline{a} \text{ and } \hat{c} \leq \psi(y_j + r\underline{a}) \\ -\infty & \text{if } a = \underline{a} \text{ and } \hat{c} \in (\psi(y_j + r\underline{a}), y_j + r\underline{a}) \\ u(\hat{c}) & \text{if } a = \underline{a} \text{ and } \hat{c} \geq y_j + r\underline{a} \end{cases} \tag{19}$$

The $\hat{u}$ utility function can be split into two sub-cases: a case where the borrowing constraint does not bind, and a case where it does. When the constraint does not bind (the first two lines), utility is given by the $\hat{u}^+(\hat{c})$ function.[31] The constraint binds when $a = \underline{a}$ and $\hat{c} = y_j + r\underline{a}$ (the fourth line). In this case the $\hat{u}$ utility function is equal to the standard CRRA utility function $u(c)$. Since $u(\hat{c}) > \hat{u}^+(\hat{c})$, the $\hat{u}$ agent can obtain a "utility boost" at $a = \underline{a}$ by setting $\hat{c} = y_j + r\underline{a}$.

The $\hat{u}$ agent earns $-\infty$ utility whenever $a = \underline{a}$ and $\hat{c} \in (\psi(y_j + r\underline{a}), y_j + r\underline{a})$. This is an off-equilibrium condition that forces the $\hat{u}$ agent to make a choice at $\underline{a}$. The $\hat{u}$ agent can either set $\hat{c}_j(\underline{a}) \leq \psi(y_j + r\underline{a})$ or $\hat{c}_j(\underline{a}) = y_j + r\underline{a}$. The former choice yields utility $\hat{u}^+$ but allows the agent to save away from the constraint. The latter choice earns the "utility boost," but requires the agent to remain at $\underline{a}$.

Utility function $\hat{u}$ is reverse-engineered to ensure that the value function of the $\hat{u}$ agent, denoted $\hat{v}_j(a)$, is equivalent to the continuation-value function $v_j(a)$ of the IG agent. To understand how the construction of $\hat{u}$ produces this value function equivalence, consider first the case where the $\hat{u}$ agent is unconstrained (either $a > \underline{a}$ or $\hat{c}_j(\underline{a}) \leq \psi(y_j + r\underline{a})$). Since the $\hat{u}$ agent is dynamically consistent while the IG agent is not, the $\hat{u}$ agent's utility must be adjusted downward to ensure that $\hat{v}_j(a) = v_j(a)$. This is why $\hat{u}^+(x) < u(x)$. Next, consider the case where the $\hat{u}$ agent is constrained. The $\hat{u}$ agent's utility no longer needs to be adjusted downward, because the $\hat{u}$ agent remains constrained at $\underline{a}$. In this case, $\hat{u}_j(y_j + r\underline{a}, \underline{a}) = u(y_j + r\underline{a})$.

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[31]In continuous time, the constraint will never bind when $a > \underline{a}$. Additionally, when $a = \underline{a}$ the constraint will not bind whenever $\hat{c} < y_j + r\underline{a}$.

Recovering the IG Equilibrium from the $\hat{u}$ Agent. The following propositions formalize the tractability that the reverse-engineered $\hat{u}$ agent provides.

Proposition 10. *Let $\hat{v}_j(a)$ denote the value function of the $\hat{u}$ agent:*

$$\hat{v}_j(a) = \max_{\{\hat{c}_s\}} \; \mathbb{E}_t \int_t^\infty e^{-\rho(s-t)} \hat{u}_j(\hat{c}_s, a_s) ds.$$

$v_j(a)$ is the continuation-value function of the IG agent if and only if $v_j(a)$ is the value function of the $\hat{u}$ agent.

Proposition 10 shows that the optimization problem of the dynamically inconsistent IG agent can be recast as a dynamically consistent optimization problem. This property is used to prove the uniqueness of the IG agent's value function (Proposition 1).

The IG agent's consumption can also be characterized relative to the $\hat{u}$ agent.

Proposition 11. *At all points where $v_j(a)$ is locally differentiable in a, the IG agent's consumption function can be characterized relative to $\hat{u}$ agent.*

1. *For $a > \underline{a}$:*

$$c_j(a) = \frac{1}{\psi}\hat{c}_j(a) \tag{20}$$

2. *For $a = \underline{a}$:*

$$c_j(a) = \begin{cases} \frac{1}{\psi}\hat{c}_j(\underline{a}) & if & \hat{c}_j(\underline{a}) \le \psi(y_j + r\underline{a}) \\ \hat{c}_j(\underline{a}) & if & \hat{c}_j(\underline{a}) = y_j + r\underline{a} \end{cases} \tag{21}$$

For all $a > \underline{a}$, equation (20) shows that the IG agent's consumption function is simply a scaled-up version of the $\hat{u}$ agent's consumption.

Understanding the $\hat{u}$ Agent. Propositions (10) and (11) show that the IG agent's behavior can be related to the behavior of the time-consistent $\hat{u}$ agent. But, the $\hat{u}$ agent is a reverse-engineered apparatus with a non-standard utility function. Accordingly, the key to

understanding the $\hat{u}$ agent is to note when the $\hat{u}$ agent does, and does not, behave identically to the standard exponential agent.

Remark 12. *If the borrowing constraint never binds for any income state j (e.g., $\underline{a} = \frac{-y_1}{r}$) then the $\hat{u}$ agent behaves identically to the standard exponential agent. In other words, $\hat{c}_j(a) = \check{c}_j(a)$. This is due to the fact that $\hat{u}^+(\hat{c})$ is a positive affine transformation of $u(c)$. When the constraint never binds then $\hat{u}_j(\hat{c}, a) = \hat{u}^+(\hat{c})$ for all (a, j).*

When the borrowing constraint does bind for some income state j, the $\hat{u}$ agent no longer behaves identically to the standard exponential agent. This is because the $\hat{u}$ agent receives a "utility boost" at the constraint. This utility boost will cause the $\hat{u}$ agent to consume more than the standard exponential agent near $\underline{a}$.

The first part of Remark 12 is critical for the results presented in Section 5 below. It implies that the intrapersonal equilibrium of the IG agent can be characterized directly from the policy functions of a standard exponential agent whenever the borrowing constraint does not bind in equilibrium.

5   Examining Present Bias in an Extended Model: High-Interest Borrowing, Illiquid Assets, and Naivete

Section 3 studies a benchmark heterogeneous-agent model with present-biased consumers. The purpose of Section 3 is to illustrate the effect of present bias in a workhorse incomplete markets model that is well-understood by economists. I now present three extensions to the demand side of the economy. First, I introduce a flexible "soft borrowing constraint." Second, I introduce an illiquid asset. Third, I allow for the agent to have incorrect beliefs about their self-control problems (i.e., naivete). All extensions in this section are presented cumulatively.

This section is conducted in partial equilibrium. I maintain the assumption throughout that the consumer is a price-taker with prices exogenously specified.

5.1 Extension 1: Soft Borrowing Constraints

To begin, I extend the benchmark model with a wealth-varying interest rate, denoted $\varsigma(a)$. $\varsigma(a)$ is defined as follows:

$$\varsigma(a) = \begin{cases} r & \text{if } a \geq 0 \\ \Gamma(a) & \text{if } a < 0 \end{cases}. \tag{22}$$

Function $\Gamma(a)$ is defined on $a \leq 0$, and is subject to the restrictions that $\Gamma(a)$ is continuously differentiable, $\Gamma'(a) \leq 0$, and $\Gamma(0) \geq r$. The natural borrowing limit is defined implicitly by $\underline{a} = \frac{-y_1}{\varsigma(\underline{a})}$.

Interest rate $\varsigma(a)$ generalizes the benchmark model in Section 3.1.[32] The benchmark model features a constant interest rate, but this is inconsistent with the data. For example, Zinman (2015) estimates that 75% of households face a borrowing cost that exceeds the risk-free rate. $\Gamma(a)$ allows for a wedge between the interest rate on borrowing versus saving, and allows for the interest rate to continue rising as the agent's debt level increases.

When $\Gamma(0) > r$, equation (22) introduces what is known as a "soft borrowing constraint." The soft constraint discourages borrowing by setting a higher interest rate for borrowing than for saving. This is in contrast to a hard borrowing constraint, such as $a \geq \underline{a}$, which completely excludes any borrowing beyond $\underline{a}$.

For the theoretical results in this section I will often assume that the borrowing constraint does not bind in equilibrium (e.g., $\underline{a}$ is the natural borrowing limit). The elimination of hard borrowing limits is not particularly restrictive given the general specification of interest rates in equation (22). $\Gamma(a)$ can become arbitrarily large as a becomes more negative. Little realism is likely to be lost by eliminating hard borrowing constraints and replacing them with arbitrarily large borrowing rates. Indeed, the latter assumption is realistic — particularly given recent financial innovations — so long as the consumer always has a credit card, payday lender, pawn shop, or loan shark that they can use to borrow an additional dollar.

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[32]The original model with a constant interest rate is recovered by setting $\Gamma(a) = r$ for all $a < 0$.

Present Bias and High-Interest Borrowing. When $\beta = 1$, soft constraints can generate a build-up of agents with exactly zero liquid wealth (e.g., Kaplan and Violante, 2014; Kaplan et al., 2018; Achdou et al., 2020). When $\beta < 1$, soft constraints no longer prevent borrowing. To show this result in a simple environment, I assume here that income is deterministic, with $y_t = y > 0$ for all t. I also assume that borrowing is allowed up to the natural borrowing limit.

Proposition 13. *Assume income is deterministic, with $y > 0$. Assume $\underline{a}$ is the natural borrowing limit. Set $r < \frac{\rho}{\beta}$ so that the IG agent dissaves for $a > 0$. Regardless of $\Gamma(a)$, the IG agent chooses to accumulate debt at $a = 0$ by setting $s(0) < 0$.*

Proposition 13 provides the stark result that the agent will always choose to borrow some amount when $\beta < 1$, regardless of how large the interest rate on borrowing is. This is not to say that the IG agent will borrow a lot, but they will accumulate some debt regardless of the interest rate on borrowing. Intuitively, debt service payments of $\varsigma(a)a$ are not very onerous when debt levels are small. At $a = 0$, the IG agent is always willing to accumulate some amount of debt for future selves to service.[33]

Proposition 13 shows that present bias can produce the high-interest borrowing that is observed empirically (e.g., Laibson et al., 2003; Zinman, 2015). It is consistent with a wide range of empirical and structural research documenting that present-biased households are more likely to borrow in the form of costly unsecured debt, such as credit cards and payday loans (e.g., Laibson et al., 2003; Meier and Sprenger, 2010; Skiba and Tobacman, 2018; Allcott et al., 2020; Kuchler and Pagel, 2020). Proposition 13 also provides theoretical justification for the use of credit card borrowing data to identify β in structural models (Laibson et al., 2020a,b).

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[33]More formally, this proof uses Proposition 16 below, which implies that the sophisticated IG agent always consumes $\frac{1}{\psi}$ times the standard exponential agent, where $\frac{1}{\psi} > 1$ when $\beta < 1$. For example, if the standard exponential agent sets $\check{c}(0) = y$, the IG agent sets $c(0) = \frac{1}{\psi}y > y$. This also implies that for a given interest rate schedule the amount borrowed is continuous in β, with the amount borrowed converging to 0 as $\beta \to 1$.

5.2 Extension 2: Liquid and Illiquid Assets

I now extend the model again to feature two assets: a liquid asset and an illiquid asset. Illiquid assets — such as houses, 401(k) and IRA savings, and social security claims — are an important source of wealth accumulation for most American households (Angeletos et al., 2001; Campbell, 2006). State of the art incomplete market models often feature both a liquid asset and an illiquid asset in order to generate a mass of wealthy hand-to-mouth agents (Kaplan and Violante, 2014; Kaplan et al., 2018). Two-asset models of this type are also an important area of research on present bias. Starting from the seminal papers of Strotz (1956) and Laibson (1997), much this literature argues that present-biased agents seek out illiquid assets as a commitment against overconsumption (Laibson et al., 1998; Amador et al., 2006; Beshears et al., 2020). This research has also been influential in the policy arena, leading to policy proposals aimed at encouraging savings through the use of illiquid assets.

In contrast to this research, this section presents an irrelevance result showing that present bias does not necessarily affect the demand for illiquid assets. Under IG preferences, the policy functions of IG agents can be compared directly to the policy functions of agents without present bias (see Section 4). I use this property to evaluate the effect of $\beta < 1$ on the demand for illiquid assets.

The extended model with an illiquid asset is similar to the household side of the HANK model of Kaplan et al. (2018). I continue to denote liquid wealth by a_t. Households can also hold wealth in an illiquid asset ζ_t. The illiquid asset has return r^ζ and volatility σ^ζ.[34] Contributions to and withdrawals from ζ are subject to a transaction cost.

I continue to assume a two-state income process for simplicity. The model and theoretical results can be extended to arbitrary Poisson income processes. In this extended model, the dynamic budget constraint becomes:

$$da_t = (y_t + \varsigma(a_t)a_t - d_t - \chi(d_t, \zeta_t) - c_t)dt \tag{23}$$

$$\frac{d\zeta_t}{\zeta_t} = \left(r^\zeta + \frac{d_t}{\zeta_t}\right)dt + \sigma^\zeta dZ_t, \tag{24}$$

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[34]r^ζ and σ^ζ can be time varying. Because this requires additional state variables, it is suppressed for notational simplicity.

subject to the restrictions that $a_t \geq \underline{a}$ and $\zeta_t \geq 0$. d_t is the household's rate of deposits (or withdrawals) into the illiquid asset, and Z_t is a standard Brownian motion. Deposits are subject to a transaction cost of $\chi(d_t, \zeta_t)$.

I assume that transaction cost function $\chi(d, \zeta)$ is everywhere differentiable with $\chi(0, \zeta) = 0$, $\chi(d, \zeta) \geq 0$, and $\frac{\partial^2 \chi(d,\zeta)}{\partial d^2} > 0$.[35] Two relevant benchmarks are: (i) perfect liquidity, with no transaction costs on asset ζ; and (ii) a 401(k)/IRA plan, with no cost to contributions and a constant 10% penalty on withdrawals. While these two benchmarks do not satisfy the restrictions on χ, they can be approximated arbitrary closely by allowable functions (i.e., they are limit points).

The borrowing constraint is more tricky in this model with liquid and illiquid assets. A simple way to deal with the constraint is to assume that the agent is not allowed to deposit nor withdraw from the illiquid account at $\underline{a}$. This is a minimal restriction, because an agent with liquid wealth of $a_t = \underline{a}$ can choose to set $c_t = y_t + \varsigma(\underline{a})\underline{a} - \epsilon$ for a single instant in order to move away from $\underline{a}$ and regain access to the illiquid asset.

The agent's problem now features two choice variables: deposit rate $d_j(a, \zeta)$ and consumption rate $c_j(a, \zeta)$. The IG agent's intrapersonal equilibrium is given by the following Bellman equation:

$$\rho v_j(a, \zeta) = u(c_j(a, \zeta)) + \frac{\partial v_j(a, \zeta)}{\partial a}(y_j + \varsigma(a)a - d_j(a, \zeta) - \chi(d_j(a, \zeta), \zeta) - c_j(a, \zeta))$$
$$+ \frac{\partial v_j(a, \zeta)}{\partial \zeta}\left(\zeta r^\zeta + d_j(a, \zeta)\right) + \lambda_j(v_{-j}(a, \zeta) - v_j(a, \zeta)) + \frac{1}{2}\frac{\partial^2 v_j(a, \zeta)}{\partial \zeta^2}(\zeta \sigma^\zeta)^2,$$

(25)

subject to the first-order conditions

$$u'(c_j(a, \zeta)) = \begin{cases} \beta \frac{\partial v_j(a,\zeta)}{\partial a} & \text{if } a > \underline{a} \\ \max\{\beta \frac{\partial v_j(a,\zeta)}{\partial a}, u'\left(y_j + \varsigma(\underline{a})\underline{a}\right)\} & \text{if } a = \underline{a} \end{cases},$$

(26)

$$\chi_d(d_j(a, \zeta), \zeta) = \begin{cases} \frac{\frac{\partial v_j(a,\zeta)}{\partial \zeta}}{\frac{\partial v_j(a,\zeta)}{\partial a}} - 1 & \text{if } a > \underline{a} \\ 0 & \text{if } a = \underline{a} \end{cases}.$$

(27)

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[35]Convexity prevents ζ_t from ever "jumping." Results are unchanged if $\chi(d, \zeta)$ features a kink at $d = 0$, which would produce an inaction region. See Kaplan et al. (2018) for details.

Equation (25) is similar to equation (7), but augmented to include the illiquid asset. Equation (26) is a slight extension of equation (8).

Equation (27) defines the asset allocation decision of $d_j(a, \zeta)$. For $a > \underline{a}$ the IG agent chooses $d_j(a, \zeta)$ to equate the marginal value of illiquid wealth to the marginal value of liquid wealth, adjusted for transaction costs: $\frac{\partial w_j(a,\zeta)}{\partial \zeta} = \frac{\partial w_j(a,\zeta)}{\partial a}(1 + \chi_d(d_j(a, \zeta), \zeta))$. Since $w_j(a, \zeta) = \beta v_j(a, \zeta)$, the multiplicative β term cancels from each side. Rearranging gives the top row of equation (27). The bottom row of equation (27) imposes the restriction that the agent cannot adjust ζ_t when $a_t = \underline{a}$.

Comparing the consumption decision (equation (26)) to the asset allocation decision (equation (27)), the key difference between the two decisions is that the β discount factor only has a direct effect on the consumption decision. Intuitively, β does not directly impact the asset allocation decision because this decision only affects the consumption of future selves. Independent of the current self's asset allocation decision, the current (instantaneous) self always has enough liquid wealth to fund current consumption whenever $a \geq \underline{a}$.

Demand for Illiquid Assets: Irrelevance of β. This extension with durables continues to feature a unique value function. Additionally, when the liquidity constraint does not bind in equilibrium then present bias has no effect on the demand for illiquid assets. This is formalized below.

Proposition 14. *The IG agent's intrapersonal equilibrium has a unique value function. When $\underline{a}$ is the natural borrowing limit, asset allocation policy function $d_j(a, \zeta)$ is independent of β.*

This irrelevance result arises in this class of two-asset models because the liquid asset eliminates any commitment properties of the illiquid asset. The agent never needs the illiquid asset in order to finance current consumption — they can always adjust their holdings of the liquid asset instead. Since the illiquid asset cannot be used to limit overconsumption, β has no effect on the policy function $d_j(a, \zeta)$.

Proposition 14 presents a general model in which present bias does not affect the demand for illiquid assets. This finding is consistent with the emerging empirical evidence that illiquid retirement-savings plans are not particularly effective savings devices. For example, Argento

et al. (2015) find that 30-40% of deposits into retirement accounts subsequently leak out before retirement.[36] Media coverage of the "retirement crisis" paints a similar picture of the inadequacy of retirement savings.

From a positive perspective, Proposition 14 speaks to a puzzle that asks why present-biased agents don't use commitment devices (Laibson, 2015; Bernheim and Taubinsky, 2018). This literature has concluded that commitment is often hampered by an important tradeoff between commitment and flexibility (Amador et al., 2006; Laibson, 2015; Carrera et al., 2020). Proposition 14 suggests a complementary explanation: it is hard to design devices that can even generate commitment in the first place. In this model, asset illiquidity provides no commitment properties because the agent has another margin (the liquid asset) that they can adjust in order to overconsume. Extending this intuition more generally, designing commitment devices is like playing a game of Whack-a-Mole. There are always margins that can be adjusted to bring utility forward in time, ranging from the consumption of unhealthy food to decreasing exercise to staying up too late. Unless a commitment device can block all of these sources of temptation, there is no reason for the agent to choose a commitment device that serves only to limit flexibility.

Proposition 14 also has normative implications. Illiquid retirement accounts are used worldwide to promote retirement savings (Beshears et al., 2015). Though cast in a stylized environment, Proposition 14 provides a crisp result highlighting the ineffectiveness of policies utilizing asset illiquidity to promote the savings of present-biased consumers.[37] Whack-a-Mole style interventions will not produce useful commitment devices, so long as the consumer can adjust on some other margin in order to bring utility forward in time. Additional details on welfare are discussed in Section 5.4.

Three caveats to Proposition 14 are necessary. First, though Proposition 14 allows for a flexible soft borrowing constraint it does still rely on the assumption that there are no hard borrowing constraints. Alternatively, models showing that present-biased agents demand illiquid assets also feature hard borrowing constraints. This difference highlights that com-

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[36]See also Choukhmane (2019), who documents that workers who are automatically enrolled in their employer's retirement savings plan adopt lower savings rates later in life.

[37]Proposition 14 says nothing about whether or not present-biased agents will save more when assets are illiquid. It only says that β does not have an independent effect on the asset allocation decision.

mitment does not come from the *illiquid* asset per se, but rather from the hard borrowing constraint on the *liquid* asset.[38] Proposition 14 may also break down under less restrictive equilibrium refinements. For example, non-Markov equilibria may feature "pseudo hard constraints" such as mental accounts (Thaler, 1985) and personal rules (Ainslie, 1992; Bernheim et al., 2015), which can reintroduce the types of notches in the consumer's budget constraint that interact with present bias to produce non-irrelevance. Olafsson and Pagel (2018) provide empirical evidence of this sort of behavior.

Second, the illiquid asset modeled in this section is best thought of as an illiquid financial asset, such as a defined-contribution pension. My results emphasize the inability for financial illiquidity to reduce overconsumption. Durables like houses and automobiles are another class of illiquid assets. Unlike financial assets, durables also provide direct consumption utility. The extent to which Proposition 14 holds with durables depends on how durables enter into the consumer's utility function.[39]

Third, Proposition 14 assumes a constant value of long-run discount rate ρ. As shown in Section 3.3, models that are calibrated to match observables will typically feature a lower ρ value when $\beta < 1$.[40] A lower calibration of ρ can certainly increase the agent's demand for illiquid assets. However, this should be recognized as an indirect effect of present bias.

5.3   Extension 3: Naivete

To this point I have assumed that the agent is fully aware of their present bias. This section generalizes the model to allow for naivete, where the agent is either partially or fully unaware of their future self-control problems (O'Donoghue and Rabin, 1999, 2001).

As above, let β denote the agent's true short-run discount factor. To extend the model to naivete, let β^E denote the short-run discount factor that the current self expects all future selves to have. Sophistication is defined as $\beta^E = \beta$, partial naivete is defined as $\beta^E \in (\beta, 1)$, and full naivete is defined as $\beta^E = 1$. The agent's expected continuation-value function

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[38]Hard constraints prevent constrained agents from freely financing consumption with the liquid asset (see equation (8)). In models with hard borrowing constraints, present-biased agents may seek to hold wealth in illiquid assets in order to put future selves at the constraint, thereby restricting future selves' consumption.

[39]Proposition 14 continues to hold under the assumption that ζ is a durable that enters utility separably.

[40]See also Laibson et al. (2020a) and Laibson et al. (2020b) for richer models that produce a similar result.

becomes:

$$v_j^E(a, \zeta) = \mathbb{E}_t \left[\int_t^\infty e^{-\rho(s-t)} u(c_s^E) ds \right],$$

where c^E denotes the consumption rate that the agent would adopt if they were sophisticated with short-run discount factor β^E. The perceived current-value function is given by:

$$w_j^E(a, \zeta) = \beta v_j^E(a, \zeta).$$

Note that the wedge between "now" and "later" is still given by true short-run discount factor β.

The intrapersonal equilibrium of Section 5.2 can be generalized to allow for naivete. Under naivete, the IG agent's expected continuation-value function v^E is given by:

$$\rho v_j^E(a, \zeta) = u(c_j^E(a, \zeta)) + \frac{\partial v_j^E(a, \zeta)}{\partial a}(y_j + \varsigma(a)a - d_j^E(a, \zeta) - \chi(d_j^E(a, \zeta), \zeta) - c_j^E(a, \zeta))$$
$$+ \frac{\partial v_j^E(a, \zeta)}{\partial \zeta}\left(\zeta r^\zeta + d_j^E(a, \zeta)\right) + \lambda_j(v_{-j}^E(a, \zeta) - v_j^E(a, \zeta)) + \frac{1}{2}\frac{\partial^2 v_j^E(a, \zeta)}{\partial \zeta^2}(\zeta \sigma^\zeta)^2,$$
$$(28)$$

subject to the first-order conditions

$$u'(c_j^E(a, \zeta)) = \begin{cases} \beta^E \frac{\partial v_j^E(a, \zeta)}{\partial a} & \text{if } a > \underline{a} \\ \max\{\beta^E \frac{\partial v_j^E(\underline{a}, \zeta)}{\partial a}, u'\left(y_j + \varsigma(\underline{a})\underline{a}\right)\} & \text{if } a = \underline{a} \end{cases}, \qquad (29)$$

$$\chi_d(d_j^E(a, \zeta), \zeta) = \begin{cases} \frac{\frac{\partial v_j^E(a, \zeta)}{\partial \zeta}}{\frac{\partial v_j^E(a, \zeta)}{\partial a}} - 1 & \text{if } a > \underline{a} \\ 0 & \text{if } a = \underline{a} \end{cases}. \qquad (30)$$

Equations (28) – (30) are identical to (25) – (27) except that the true short-run discount factor β is replaced by the perceived discount factor β^E. This implies that the perceived continuation-value function $v_j^E(a, \zeta)$ is still unique under naivete:

Corollary 15. *The expected value function $v_j^E(a, \zeta)$ is unique.*

Proof. Uniqueness of v^E follows directly from Proposition 14. □

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Because the naive agent does not necessarily behave according to their expectations, equations (28) – (30) only pin down the expected continuation-value function v_t^E. The actual consumption and asset allocation decisions of the naive agent are given by:

$$u'(c_j(a,\zeta)) = \begin{cases} \beta \frac{\partial v_j^E(a,\zeta)}{\partial a} & \text{if } a > \underline{a} \\ \max\{\beta \frac{\partial v_j^E(\underline{a},\zeta)}{\partial a}, u'(y_j + \varsigma(\underline{a})\underline{a})\} & \text{if } a = \underline{a} \end{cases}, \tag{31}$$

$$\chi_d(d_j(a,\zeta),\zeta) = \begin{cases} \frac{\frac{\partial v_j^E(a,\zeta)}{\partial \zeta}}{\frac{\partial v_j^E(a,\zeta)}{\partial a}} - 1 & \text{if } a > \underline{a} \\ 0 & \text{if } a = \underline{a} \end{cases}. \tag{32}$$

Comparing the agent's actual behavior to their perceived behavior, naivete only creates incorrect expectations about the consumption decision. The actual consumption decision in (31) is determined with the true short-run discount factor β, while the perceived consumption decision in (29) depends on β^E. Alternatively, the actual asset allocation decision is equivalent to the perceived asset allocation decision because the asset allocation decision only affects the consumption of future selves.

The Consumption of Present-Biased Consumers. The extended model now allows for stochastic income, soft borrowing constraints, liquid and illiquid wealth, and naivete. In this rich model, IG preferences can be used to provide a simple closed-form expression that characterizes the effect of present bias on consumption-saving decisions.

Proposition 16. *Let β denote the agent's true short-run discount factor, and let $\beta^E \in [\beta, 1]$ denote the agent's perceived short-run discount factor. Let $\psi^E = \frac{\gamma - (1 - \beta^E)}{\gamma}$. Assume that $\underline{a}$ is the natural borrowing limit. Relative to a standard exponential agent, the consumption of the potentially naive IG agent is given by:*

$$c_j(a,\zeta) = \left(\frac{\beta^E}{\beta}\right)^{\frac{1}{\gamma}} \frac{1}{\psi^E} \times \check{c}_j(a,\zeta). \tag{33}$$

Equation (33) simplifies even further in two special cases of particular interest. Under sophistication, the IG agent consumes $\frac{1}{\psi}$ more than the standard exponential agent. Under full naivete, the IG agent consumes $\beta^{-\frac{1}{\gamma}}$ more than the standard exponential agent.

Proposition 16 can also be used to compare the consumption of sophisticates versus naifs, even in this complex economic environment that cannot be solved in closed form. Again, the critical assumption is that the borrowing constraint $\underline{a}$ does not bind in equilibrium.

Corollary 17. *Assume that $\underline{a}$ is the natural borrowing limit. If a sophisticate with $\beta^E = \beta$ consumes c, a naif with $\beta^E \in (\beta, 1]$ will consume $\left(\frac{\beta^E}{\beta}\right)^{\frac{1}{\gamma}} \frac{\gamma-(1-\beta)}{\gamma-(1-\beta^E)} \times c$. Consumption is increasing in naivete when $\gamma > 1$ and decreasing in naivete when $\gamma < 1$.*

Proof. The standard exponential agent's consumption function of $\check{c}_j(a, \zeta)$ is independent of β and β^E. This corollary then follows from equation (33). $\qquad \square$

In simple environments featuring linear consumption functions, it is known that sophisticates and naifs adopt the same equilibrium consumption function when $\gamma = 1$ (e.g., Tobacman, 2007). Corollary 17 shows that this result continues to hold in more general consumption-saving models, so long as $\underline{a}$ does not bind in equilibrium. Intuitively, naivete introduces two offsetting effects. One the one hand, the naif is more willing to save because the naif trusts the consumption decisions of future selves. On the other hand, the naif is less willing to save because they believe that their future selves will save more on their own. These two effects exactly offset when $\gamma = 1$.

Taking this equivalence between sophisticates and naifs one step further, Proposition 16 also implies that there is also an *observational* equivalence between sophisticates and naifs.

Corollary 18. *Assume that $\underline{a}$ is the natural borrowing limit. A sophisticated agent with short-run discount factor β will consume identically to a naive agent with a perceived short-run discount factor of $\beta^E \in [\beta, 1]$ and a true short-run discount factor of $\beta' = \beta^E \left[\frac{\gamma-(1-\beta)}{\gamma-(1-\beta^E)}\right]^\gamma$.*

Proof. Again, since the standard exponential agent's consumption function of $\check{c}_j(a, \zeta)$ is independent of β and β^E, this corollary follows from equation (33). $\qquad \square$

A takeaway from Corollary 18 is that sophistication versus naivete cannot be easily identified from consumption choices. Though Corollary 18 does not hold in models featuring binding constraints, differences in consumption that are driven by binding constraints are likely not the central predictions that form the basis of robust identification. The consumption of sophisticates may also differ from naifs in discrete-time models with large time-steps,

but one should be cautious about using time-steps that are psychologically inappropriate. Additionally, Proposition 14 above (and Corollary 21 below) shows that researchers may also struggle to use demand for commitment to identify naivete. Instead, it is more likely that identification can be found by evaluating data on procrastination (O'Donoghue and Rabin, 1999), contract choice (DellaVigna and Malmendier, 2004; Gabaix and Laibson, 2006; Heidhues and Kőszegi, 2010), and the accuracy of household budgeting plans (Augenblick and Rabin, 2019; Allcott et al., 2020; Kuchler and Pagel, 2020), as these sorts of decisions follow directly from agents' misperception of their future actions.

Euler Equation. For intuition on the effect of naivete, Appendix D solves the Aiyagari-Bewley-Huggett model of Section 3.3 under the assumption of complete naivete. Results are qualitatively similar. The case with naivete produces slightly less overconsumption near $\underline{a}$, because naivete eliminates the interaction between present bias and finite planning horizons.

For additional intuition, in the benchmark model of Section 3.1 the Euler equation of Corollary 3 can be extended to the case of naivete (see Tobacman (2007) for a discrete-time analysis, and Laibson et al. (2020b) for an application of this result).

Corollary 19. *At all points where $c_j(a)$ is locally differentiable in a, the consumption function $c_j(a)$ satisfies the following Euler equation:*

$$\frac{\mathbb{E}_t \left(du'(c_j(a_t))/dt \right)}{u'(c_j(a_t))} = \left[\rho + \frac{\partial c_j(a_t)}{\partial a} \left((1 - \beta^E) \left(\frac{\beta}{\beta^E} \right)^{\frac{1}{\gamma}} + \gamma \left(1 - \left(\frac{\beta}{\beta^E} \right)^{\frac{1}{\gamma}} \right) \right) \right] - r. \quad (34)$$

Corollary 19 simplifies in three special cases: complete sophistication ($\beta^E = \beta$), complete naivete ($\beta^E = 1$), and log utility ($\gamma = 1$). With complete sophistication or $\gamma = 1$, the Euler equation in (13) is recovered. With complete naivete, the Euler equation is:

$$\frac{\mathbb{E}_t \left(du'(c_j(a_t))/dt \right)}{u'(c_j(a_t))} = \left[\rho + \gamma \left(1 - \beta \right)^{\frac{1}{\gamma}} c_j'(a_t) \right] - r$$

As in the case with sophistication, the effective discount rate in brackets varies with the instantaneous MPC of the naive IG agent. Under naivete, overconsumption has a larger effect on the growth rate of marginal utility when consumption itself is sensitive to a_t (i.e.,

$c'_j(a_t)$ is large). In the case of partial naivete, the Euler equation reflects both sophistication-driven and naivete-driven motives for a state-dependent effective discount rate.

5.4 Putting It All Together: Present Bias and Welfare

Welfare analyses can be difficult in models with dynamically inconsistent preferences because these preferences do not typically feature a single welfare criterion. In models with present bias, the common approach is to adopt the "long-run" view in which policymakers seek to maximize the continuation-value function v_t. However, this approach ignores the preferences of each individual self. This may be unsatisfactory if, for example, the current self is better able to evaluate their immediate preferences than a distant self (Bernheim and Rangel, 2009; Bernheim and Taubinsky, 2018).

Unlike discrete-time models with large time-steps, continuous-time IG preferences feature a single welfare criterion (Harris and Laibson, 2013). This is an important, though relatively unrecognized, feature of IG preferences. This property arises because each self lives for only an instant, and therefore composes a non-measurable part of the overall value function. More formally, the current self wants to adopt policies that maximize current-value function w_t. Since $w_t = \beta v_t$, any policy that maximizes v_t will also maximize w_t.[41] This single welfare criterion property, combined with the modeling tractability provided by IG preferences, means that IG preferences are well suited for the analysis of policy in complex economic environments.

Building on this insight, I show that IG preferences allow for a simple characterization of the welfare cost of present bias. I present this welfare result in the context of the two-asset model with naivete of Section 5.3. However, this welfare characterization will continue to hold in more general environments, as long as $\underline{a}$ does not bind in equilibrium.

In order to present a welfare metric that applies in the general consumption-saving environment that I am studying, I consider the following experiment. Assume that there

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[41]Though the single welfare criterion property only holds exactly in continuous time, it is robust to discrete time with psychologically appropriate time-steps. For example, consider a discrete-time model where each self lives for one day. Let $\delta = \exp\left(\frac{-\rho}{365}\right)$. Assume that each self consumes a constant amount, $\bar{c}$, in each period. Then, the current-value function is $w = u(\bar{c}) + \beta\frac{\delta u(\bar{c})}{1-\delta}$. Each one-day self composes a share $\frac{1}{1+\beta\frac{\delta}{1-\delta}}$ of the total current-value function. Under the calibration in Section 3.3 ($\rho = 2.5\%$ and $\beta = 0.75$), this means that the current self composes only 0.009% of the total value function.

exists a perfect commitment device that forces all future selves to behave as if $\beta = 1$. This commitment device comes at the cost of a perpetual consumption tax of τ. The following Proposition characterizes the welfare loss of present bias in terms of consumption tax τ.

Proposition 20. *Assume $\underline{a}$ is the natural borrowing limit. Let $\alpha = \psi^E \left(\frac{\beta}{\beta^E} \right)^{\frac{1}{\gamma}}$. The welfare cost of present bias is equivalent to a perpetual consumption tax of:*[42]

$$\tau = 1 - \left(\frac{\alpha^\gamma}{1 - \gamma + \gamma\alpha} \right)^{\frac{1}{1-\gamma}}. \tag{35}$$

τ is decreasing in β, and τ is increasing in β^E if and only if $\gamma > 1$.

    Proposition 20 is powerful because it is very general. It holds in the benchmark model of Section 3.1, the model with soft constraints as in equation (22), the model with illiquid assets of Section 5.2, and across varying levels of naivete. It will also hold for richer income processes than the two-state process I assume here. As long as $\underline{a}$ does not bind in equilibrium, the welfare cost of present bias can be represented as a consumption tax of size τ.

    The fact that such a simple formula characterizes the welfare cost of present bias across a general class of models may seem surprising. IG preferences make this welfare characterization possible. The proof relies on the fact that the value function of the IG agent can always be recast as the value function of an exponential agent with a modified utility function. When the agent is sophisticated, the modified utility function is denoted $\hat{u}$ (see Section 4). The proof of Proposition 20 generalizes the modified utility function to allow for naivete, denoted $\hat{\hat{u}}$ (see Appendix C). When $\underline{a}$ does not bind, the proof boils down to finding the value of τ such that $\hat{\hat{u}}(x) = u((1 - \tau)x)$.[43]

Discussion.   The first takeaway from Proposition 20 is that present bias can be very costly. For example, the calibration in Section 3 features $\beta = \beta^E = 0.75$ and $\gamma = 2$. The welfare

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[42]When $\gamma = 1$, the tax is given by $\tau = 1 - \frac{\exp\left(\frac{\beta-1}{\beta}\right)}{\beta}$.

[43]It is worth noting that τ is only defined when $1 - \gamma + \gamma\alpha > 0$. This is not necessarily the case under naivete when $\gamma > 1$ and α is low. In these cases, the naif behaves so poorly that their realized value function goes to $-\infty$. Intuitively, the naif always thinks that they are only overconsuming for a single instant. When this mistake is made repeatedly, it can lead to a realized value function that is undefined (though at each point in time, the naif perceives that their value function is finite). The condition that $1 - \gamma + \gamma\alpha > 0$ can be thought of as a bound on the level of naivete that is theoretically admissible.

cost of present bias is equivalent to a 2% consumption tax. For $\beta = 0.5$ and $\beta^E = 1$, as estimated in Laibson et al. (2020a), the welfare cost of present bias is equivalent to a 17.2% consumption tax. These costs are at least an order of magnitude larger than benchmark estimates of the welfare cost of business cycles (Lucas, 1987).

One caveat to these large welfare costs is that τ gives the realized cost of present bias. If the agent is naive, they may not perceive such a large welfare cost. A naif perceives that the welfare cost of their present bias is equivalent to a tax of $\tau^E = 1 - \left(\frac{(\psi^E)^\gamma}{\beta^E} \right)^{\frac{1}{1-\gamma}}$.[44]

The second takeaway from Proposition 20 is that the welfare cost of present bias depends only on β, β^E, and γ. The welfare cost of present bias is decreasing in β, which is straightforward. The welfare cost is worse under naivete when $\gamma > 1$, and worse under sophistication when $\gamma < 1$. This follows from Corollary 17: overconsumption is exacerbated by naivete when $\gamma > 1$, and is reduced by naivete when $\gamma < 1$.

Another way to look at this second takeaway is to consider what the welfare cost of present bias does not depend on. The welfare cost of present bias is independent of liquid wealth a, illiquid wealth ζ, income state j, and market price parameters y_j, r, Γ, χ, r^ζ, and σ^ζ. Though these variables certainly affect the agent's welfare, they do not independently affect the relative welfare cost of present bias.

This intuition yields the following Proposition, which is a key policy implication of this welfare analysis.

Proposition 21. *Assume $\underline{a}$ is the natural borrowing limit. A policy intervention that alters the income process, interest rates, and/or transaction costs improves the welfare of the IG agent if and only if it improves the welfare of the standard exponential agent.*

Proof. The proof of Proposition 21 follows directly from the proof of Proposition 20, which shows that the IG agent's value function is a positive affine transformation of the standard exponential agent's value function. □

From the perspective of an individual IG consumer, Proposition 21 implies that commitment devices, such as penalty borrowing rates and asset illiquidity, will not alleviate the

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[44]This is the maximum tax that a sophisticate with discount factor β^E would accept.

negative effects of present bias. To the extent that such devices are undesirable for a time-consistent agent, they will also make the IG agent worse off. This forms the basis of the present-bias dilemma, which is presented in Section 6 below.

From the perspective of a policymaker, Proposition 21 is an irrelevance result: the policymaker does not need to consider present bias when determining whether or not a given policy is welfare improving. Instead, where present bias matters is in determining whether or not a given policy is feasible (i.e., whether it obeys a budget constraint). For example, consider a large interest rate subsidy on savings that is financed by a small consumption tax. Though such a policy may be welfare improving regardless of β and β^E, the revenue collected by the consumption tax may only be sufficient to cover the cost of the interest rate subsidy in an economy populated by present-biased consumers. This is because present-biased agents will not take full advantage of the policy, underusing the interest rate subsidy and overpaying the consumption tax. Appendix B provides a toy model.

6   The Present-Bias Dilemma

Propositions 20 and 21 show that the present-biased agent faces a critical dilemma. On the one hand, present bias can be enormously costly. On the other hand, the agent cannot use commitment devices in the form of financial penalties to curtail their present bias. The welfare cost of present bias depends only on β, β^E, and γ. Financial commitment devices, such as self-imposed asset illiquidity or penalty borrowing rates, are unable to reduce this welfare cost. Instead, as Proposition 21 highlights, these sorts of self-imposed commitments only make the IG agent worse off (because they make the standard exponential agent worse off).

Similar to Proposition 14, Proposition 21 relates to the puzzle of why present-biased agents don't use commitment devices. As Proposition 21 shows, commitment in the form of financial penalties does not improve the welfare of the IG agent. Such commitment devices only benefit the IG agent if they benefit the standard exponential agent, and a time-consistent agent will never choose to self-impose financial costs. For intuition, consider a penalty interest rate on borrowing. Though this will reduce borrowing, the borrowing that

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still occurs becomes more costly.[45]

I propose two potential solutions to this present-bias dilemma. First, consumers may have access to non-financial commitment devices. The literature on self-control has found a variety of non-financial "nudges" that help to reduce present bias, ranging from explicit planning (Beshears et al., 2016) to reminders (Karlan et al., 2016).[46] A theoretical analysis of these sorts of non-financial commitment devices is beyond the scope of this paper, and is an important area for future research.

Second, the present-bias dilemma is about the inability for any individual agent to self-impose financial commitments that alleviate their present bias. Government interventions are fundamentally different from what a consumer can self-impose. That is, the government has the ability not only to implement financial disincentives in the form of taxes, but also to redistribute tax revenue back to consumers. To show this point, Appendix B presents a simple "Eat-the-Pie" model to capture how the combination of taxes plus redistribution in the form of a savings subsidy can improve the welfare of present-biased agents. Such penalty-plus-redistribution policies could also be implemented by private institutions in addition to the government, as it is the pooling of consumers that is essential.[47] Finally, it's worth noting that the logic of the model suggests that comprehensive policies that incentivize a focus on the future, such as a savings subsidy, are likely to be more effective than tailored Whack-a-Mole style interventions that the IG agent can circumvent.

Because present bias is highly costly but also difficult for the individual consumer to correct, the present-bias dilemma highlights an important role for the government and/or private institutions in alleviating present bias. The analytical and numerical methods developed in this paper will be useful for quantitative models that evaluate the effectiveness of various policy interventions.

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[45]Proposition 16 shows that the IG agent always consumes $(\frac{\beta^E}{\beta})^{\frac{1}{\gamma}} \frac{1}{\psi^E}$ times the standard exponential agent. Thus, financial penalties do not affect the *relative* overconsumption of the IG agent.

[46]See Bernheim and Taubinsky (2018) for a review. See also Laibson (2018) for a related discussion of paternalistic policies employed by private institutions.

[47]See Laibson (1997) for a discussion of why this may be difficult for the private sector to implement. Additionally, political economy considerations are beyond the scope of this paper, but are important for understanding the extent to which governments can respond to the present bias of their constituents. Present focus can also arise in policymakers' decisions. For a small set of examples, see Persson and Svensson (1989), Alesina and Tabellini (1990), Aguiar and Amador (2011), Halac and Yared (2014), and Jackson and Yariv (2014).

7 Conclusion

This paper presents a new set of continuous-time methods for modeling the consumption-saving decisions of consumers with present bias. The first part of this paper uses these methods to solve a workhorse Aiyagari-Bewley-Huggett incomplete markets model. The second part analytically characterizes the effect of present bias on consumption and saving decisions, and provides a closed-form expression for the welfare cost of present bias.

The IG methods that I present in this paper open many avenues for future research. For macroeconomists, understanding the general equilibrium effects of present bias in state of the art heterogeneous-agent models is an important next step. The IG toolbox may also be useful in structural behavioral economics, allowing for the estimation of present-biased time preferences across heterogeneous consumers using field data and rich structural models (DellaVigna, 2018).

The key partial equilibrium result of this paper is the present-bias dilemma: present bias has large welfare costs, but agents cannot use self-imposed financial commitments such as illiquid savings accounts to improve their welfare. For behavioral economists, it is essential to continue developing non-financial commitment devices that can improve agents' self-control. For public economists, an analysis of the sorts of government interventions that can alleviate present bias, particularly in an economy populated by agents with heterogeneous preferences, is likely to prove fruitful.

References

Achdou, Yves, Jiequn Han, Jean-Michel Lasry, Pierre-Louis Lions, and Benjamin Moll, "Income and Wealth Distribution in Macroeconomics: A Continuous-Time Approach," *The Review of Economic Studies*, 2020, *Forthcoming.*

Agarwal, Sumit, Souphala Chomsisengphet, Neale Mahoney, and Johannes Stroebel, "Do Banks Pass Through Credit Expansions to Consumers Who Want to Borrow?," *The Quarterly Journal of Economics*, 2018, *133* (1), 129–190.

Aguiar, Mark and Manuel Amador, "Growth in the Shadow of Expropriation," *The Quarterly Journal of Economics*, 2011, *126* (2), 651–697.

Ainslie, George, *Picoeconomics: The Strategic Interaction of Successive Motivational States Within the Person*, Cambridge University Press, 1992.

Aiyagari, S. Rao, "Uninsured Idiosyncratic Risk and Aggregate Saving," *The Quarterly Journal of Economics*, 1994, *109* (3), 659–684.

Akerlof, George A., "Procrastination and Obedience," *American Economic Review*, 1991, *81* (2), 1–19.

Alesina, Alberto and Guido Tabellini, "A Positive Theory of Fiscal Deficits and Government Debt," *The Review of Economic Studies*, 1990, *57* (3), 403–414.

Allcott, Hunt, Benjamin B. Lockwood, and Dmitry Taubinsky, "Regressive Sin Taxes, with an Application to the Optimal Soda Tax," *The Quarterly Journal of Economics*, 2019, *134* (3), 1557–1626.

_ , **Joshua Kim, Dmitry Taubinsky, and Jonathan Zinman**, "Are High-Interest Loans Predatory? Theory and Evidence from Payday Lending," *Mimeo*, 2020.

Amador, Manuel, Iván Werning, and George-Marios Angeletos, "Commitment vs. Flexibility," *Econometrica*, 2006, *74* (2), 365–396.

Angeletos, George-Marios, David Laibson, Andrea Repetto, Jeremy Tobacman, and Stephen Weinberg, "The Hyperbolic Consumption Model: Calibration, Simulation, and Empirical Evaluation," *Journal of Economic Perspectives*, 2001, *15* (3), 47–68.

Argento, Robert, Victoria L. Bryant, and John Sabelhaus, "Early Withdrawals from Retirement Accounts During the Great Recession," *Contemporary Economic Policy*, 2015, *33* (1), 1–16.

Attanasio, Orazio, Agnes Kovacs, and Patrick Moran, "Temptation and Commitment: Understanding Hand-to-Mouth Behavior," *NBER w27944*, 2020.

Augenblick, Ned, "Short-Term Time Discounting of Unpleasant Tasks," *Mimeo*, 2018.

__ **and Matthew Rabin**, "An Experiment on Time Preference and Misprediction in Unpleasant Tasks," *The Review of Economic Studies*, 2019, *86* (3), 941–975.

__ , **Muriel Niederle, and Charles Sprenger**, "Working over Time: Dynamic Inconsistency in Real Effort Tasks," *The Quarterly Journal of Economics*, 2015, *130* (3), 1067–1115.

Ausubel, Lawrence M., "The Failure of Competition in the Credit Card Market," *American Economic Review*, 1991, *81* (1), 50–81.

Banerjee, Abhijit and Sendhil Mullainathan, "The Shape of Temptation: Implications for the Economic Lives of the Poor," *NBER w15973*, 2010.

Barles, Guy and Panagiotis E. Souganidis, "Convergence of Approximation Schemes for Fully Nonlinear Second Order Equations," *Asymptotic Analysis*, 1991, *4* (3), 271–283.

Barro, Robert J., "Ramsey Meets Laibson in the Neoclassical Growth Model," *The Quarterly Journal of Economics*, 1999, *114* (4), 1125–1152.

Benhabib, Jess and Alberto Bisin, "Skewed Wealth Distributions: Theory and Empirics," *Journal of Economic Literature*, 2018, *56* (4), 1261–91.

Bernheim, B. Douglas and Antonio Rangel, "Beyond Revealed Preference: Choice-Theoretic Foundations for Behavioral Welfare Economics," *The Quarterly Journal of Economics*, 2009, *124* (1), 51–104.

__ **and Dmitry Taubinsky**, "Behavioral Public Economics," in "Handbook of Behavioral Economics: Applications and Foundations 1," Vol. 1, Elsevier, 2018, pp. 381–516.

__ , **Debraj Ray, and Şevin Yeltekin**, "Poverty and Self-Control," *Econometrica*, 2015, *83* (5), 1877–1911.

Beshears, John, James J. Choi, Christopher Clayton, Christopher Harris, David Laibson, and Brigitte C. Madrian, "Optimal Illiquidity," *NBER w27459*, 2020.

_ , _ , **Joshua Hurwitz, David Laibson, and Brigitte C. Madrian**, "Liquidity in Retirement Savings Systems: An International Comparison," *American Economic Review*, 2015, *105* (5), 420–25.

_ , **Katherine L. Milkman, and Joshua Schwartzstein**, "Beyond Beta-Delta: The Emerging Economics of Personal Plans," *American Economic Review*, 2016, *106* (5), 430–34.

Bewley, Truman, "Stationary Monetary Equilibrium with a Continuum of Independently Fluctuating Consumers," *Contributions to Mathematical Economics in Honor of Gérard Debreu*, 1986, *79.*

Bond, Philip and Gustav Sigurdsson, "Commitment Contracts," *The Review of Economic Studies*, 2018, *85* (1), 194–222.

Bryan, Gharad, Dean Karlan, and Scott Nelson, "Commitment Devices," *Annual Review Economics*, 2010, *2* (1), 671–698.

Campbell, John Y., "Household Finance," *The Journal of Finance*, 2006, *61* (4), 1553–1604.

_ **and N. Gregory Mankiw**, "Consumption, Income, and Interest Rates: Reinterpreting the Time Series Evidence," *NBER Macroeconomics Annual*, 1989, *4*, 185–216.

Cao, Dan and Iván Werning, "Dynamic Savings Choices with Disagreements," *NBER w22007*, 2016.

_ **and** _ , "Saving and Dissaving with Hyperbolic Discounting," *Econometrica*, 2018, *86* (3), 805–857.

Carrera, Mariana, Heather Royer, Mark Stehr, Justin Sydnor, and Dmitry Taubinsky, "Who Chooses Commitment? Evidence and Welfare Implications," *Mimeo*, 2020.

Carroll, Christopher D., "Buffer-Stock Saving and the Life Cycle/Permanent Income Hypothesis," *The Quarterly Journal of Economics*, 1997, *112* (1), 1–55.

Chatterjee, Satyajit and Burcu Eyigungor, "Continuous Markov Equilibria with Quasi-Geometric Discounting," *Journal of Economic Theory*, 2016, *163*, 467–494.

Choukhmane, Taha, "Default Options and Retirement Saving Dynamics," *Mimeo*, 2019.

Cohen, Jonathan, Keith Marzilli Ericson, David Laibson, and John Myles White, "Measuring Time Preferences," *Journal of Economic Literature*, 2020, *58* (2), 299–347.

Deaton, Angus, "Saving and Liquidity Constraints," *Econometrica*, 1991, *59* (5), 1221–1248.

DellaVigna, Stefano, "Psychology and Economics: Evidence from the Field," *Journal of Economic Literature*, 2009, *47* (2), 315–72.

__ , "Structural Behavioral Economics," in "Handbook of Behavioral Economics: Applications and Foundations 1," Vol. 1, Elsevier, 2018, pp. 613–723.

__ **and M. Daniele Paserman**, "Job Search and Impatience," *Journal of Labor Economics*, 2005, *23* (3), 527–588.

__ **and Ulrike Malmendier**, "Contract Design and Self-Control: Theory and Evidence," *The Quarterly Journal of Economics*, 2004, *119* (2), 353–402.

__ **and** __ , "Paying Not to Go to the Gym," *American Economic Review*, 2006, *96* (3), 694–719.

Fagereng, Andreas, Martin Blomhoff Holm, Benjamin Moll, and Gisle Natvik, "Saving Behavior Across the Wealth Distribution: The Importance of Capital Gains," *NBER 26588*, 2019.

Farhi, Emmanuel and Xavier Gabaix, "Optimal Taxation with Behavioral Agents," *American Economic Review*, 2020, *110* (1), 298–336.

Feldstein, Martin and Jeffrey B. Liebman, "Social Security," *Handbook of Public Economics*, 2002, *4*, 2245–2324.

Floden, Martin and Jesper Lindé, "Idiosyncratic Risk in the United States and Sweden: Is There a Role for Government Insurance?," *Review of Economic Dynamics*, 2001, *4* (2), 406–437.

Frederick, Shane, George Loewenstein, and Ted O'Donoghue, "Time Discounting and Time Preference: A Critical Review," *Journal of Economic Literature*, 2002, *40* (2), 351–401.

Gabaix, Xavier and David Laibson, "Shrouded Attributes, Consumer Myopia, and Information Suppression in Competitive Markets," *The Quarterly Journal of Economics*, 2006, *121* (2), 505–540.

Galperti, Simone, "Commitment, Flexibility, and Optimal Screening of Time Inconsistency," *Econometrica*, 2015, *83* (4), 1425–1465.

Ganong, Peter and Pascal Noel, "Consumer Spending During Unemployment: Positive and Normative Implications," *American Economic Review*, 2019, *109* (7), 2383–2424.

Gathergood, John, "Self-Control, Financial Literacy and Consumer Over-Indebtedness," *Journal of Economic Psychology*, 2012, *33* (3), 590–602.

Gourinchas, Pierre-Olivier and Jonathan A. Parker, "Consumption over the Life Cycle," *Econometrica*, 2002, *70* (1), 47–89.

Gross, David B. and Nicholas S. Souleles, "Do Liquidity Constraints and Interest Rates Matter for Consumer Behavior? Evidence from Credit Card Data," *The Quarterly Journal of Economics*, 2002, *117* (1), 149–185.

Gruber, Jonathan and Botond Köszegi, "Is Addiction "Rational"? Theory and Evidence," *The Quarterly Journal of Economics*, 2001, *116* (4), 1261–1303.

Guerrieri, Veronica and Guido Lorenzoni, "Credit Crises, Precautionary Savings, and the Liquidity Trap," *The Quarterly Journal of Economics*, 2017, *132* (3), 1427–1467.

Guvenen, Fatih, "Macroeconomics with Hetereogeneity: A Practical Guide," *Economic Quarterly*, 2011, *97* (3Q), 255–326.

Halac, Marina and Pierre Yared, "Fiscal Rules and Discretion Under Persistent Shocks," *Econometrica*, 2014, *82* (5), 1557–1614.

Harris, Christopher and David Laibson, "Dynamic Choices of Hyperbolic Consumers," *Econometrica*, 2001, *69* (4), 935–957.

_ **and** _ , "Hyberbolic Discounting and Consumption," *Econometric Society Monographs*, 2003, *35*, 258–297.

_ **and** _ , "Instantaneous Gratification," *Mimeo*, 2004.

_ **and** _ , "Instantaneous Gratification," *The Quarterly Journal of Economics*, 2013, *128* (1), 205.

Heathcote, Jonathan, Kjetil Storesletten, and Giovanni L. Violante, "Quantitative Macroeconomics with Heterogeneous Households," *Annual Review of Economics*, 2009, *1* (1), 319–354.

Heidhues, Paul and Botond Kőszegi, "Exploiting Naivete About Self-Control in the Credit Market," *American Economic Review*, 2010, *100* (5), 2279–2303.

Huggett, Mark, "The Risk-Free Rate in Heterogeneous-Agent Incomplete-Insurance Economies," *Journal of Economic Dynamics and Control*, 1993, *17* (5-6), 953–969.

Imrohoroğlu, Ayşe, "Cost of Business Cycles with Indivisibilities and Liquidity Constraints," *Journal of Political Economy*, 1989, *97* (6), 1364–1383.

Jackson, Matthew O. and Leeat Yariv, "Present Bias and Collective Dynamic Choice in the Lab," *American Economic Review*, 2014, *104* (12), 4184–4204.

Kaplan, Greg and Giovanni L. Violante, "A Model of the Consumption Response to Fiscal Stimulus Payments," *Econometrica*, 2014, *82* (4), 1199–1239.

_ **and** _ , "Microeconomic Heterogeneity and Macroeconomic Shocks," *Journal of Economic Perspectives*, 2018, *32* (3), 167–94.

_ , **Benjamin Moll, and Giovanni L. Violante**, "Monetary Policy According to HANK," *American Economic Review*, 2018, *108* (3), 697–743.

Karlan, Dean, Margaret McConnell, Sendhil Mullainathan, and Jonathan Zinman, "Getting to the Top of Mind: How Reminders Increase Saving," *Management Science*, 2016, *62* (12), 3393–3411.

Kovacs, Agnes and Patrick Moran, "Breaking the Commitment Device: The Effect of Home Equity Withdrawal on Consumption, Saving, and Welfare," *Mimeo*, 2020.

Krebs, Tom, "Job Displacement Risk and the Cost of Business Cycles," *American Economic Review*, 2007, *97* (3), 664–686.

Krueger, Dirk, Kurt Mitman, and Fabrizio Perri, "Macroeconomics and Household Heterogeneity," in "Handbook of Macroeconomics," Vol. 2, Elsevier, 2016, pp. 843–921.

Krusell, Per and Anthony A. Smith, "Consumption–Savings Decisions with Quasi-Geometric Discounting," *Econometrica*, 2003, *71* (1), 365–375.

_ **and Anthony A. Smith Jr.**, "Income and Wealth Heterogeneity in the Macroeconomy," *Journal of Political Economy*, 1998, *106* (5), 867–896.

_ , **Toshihiko Mukoyama, Ayşegül Şahin, and Anthony A. Smith Jr.**, "Revisiting the Welfare Effects of Eliminating Business Cycles," *Review of Economic Dynamics*, 2009, *12* (3), 393–404.

Kuchler, Theresa and Michaela Pagel, "Sticking to Your Plan: The Role of Present Bias for Credit Card Paydown," *Journal of Financial Economics*, 2020.

Laibson, David, "Self-Control and Saving," *Mimeo*, 1994.

_ , "Golden Eggs and Hyperbolic Discounting," *The Quarterly Journal of Economics*, 1997, *112* (2), 443–478.

_ , "Life-Cycle Consumption and Hyperbolic Discount Functions," *European Economic Review*, 1998, *42* (3-5), 861–871.

_ , "Why Don't Present-Biased Agents Make Commitments?," *American Economic Review*, 2015, *105* (5), 267–72.

_ , "Private Paternalism, the Commitment Puzzle, and Model-Free Equilibrium," *American Economic Review*, 2018, *108*, 1–21.

_ , , **Sean Chanwook Lee, Peter Maxted, Andrea Repetto, and Jeremy Tobacman**, "Estimating Discount Functions with Consumption Choices Over the Lifecycle," *Mimeo*, 2020.

_ **and Peter Maxted**, "The $\beta - \delta - \Delta$ Sweet Spot," *Mimeo*, 2020.

_ , **Andrea Repetto, and Jeremy Tobacman**, "A Debt Puzzle," *Knowledge, Information, and Expectations in Modern Macroeconomics: In Honor of Edmund S. Phelps*, 2003.

Laibson, David I., "Hyperbolic Discount Functions, Undersaving, and Savings Policy," *NBER w5635*, 1996.

_ , **Andrea Repetto, and Jeremy Tobacman**, "Self-Control and Saving for Retirement," *Brookings Papers on Economic Activity*, 1998, *1998* (1), 91–196.

Laibson, David, Peter Maxted, and Benjamin Moll, "Present Bias Amplifies the Household Balance-Sheet Channels of Macroeconomic Policy," *Mimeo*, 2020.

Lockwood, Benjamin B., "Optimal Income Taxation with Present Bias," *American Economic Journal: Economic Policy*, Forthcoming.

Lucas, Robert E., *Models of Business Cycles*, Vol. 26, Basil Blackwell Oxford, 1987.

Luttmer, Erzo G.J. and Thomas Mariotti, "Subjective Discounting in an Exchange Economy," *Journal of Political Economy*, 2003, *111* (5), 959–989.

Madrian, Brigitte C. and Dennis F. Shea, "The Power of Suggestion: Inertia in 401 (k) Participation and Savings Behavior," *The Quarterly Journal of Economics*, 2001, *116* (4), 1149–1187.

Maliar, Lilia and Serguei Maliar, "The Neoclassical Growth Model with Heterogeneous Quasi-Geometric Consumers," *Journal of Money, Credit and Banking*, 2006, *38* (3), 635–654.

Maskin, Eric and Jean Tirole, "Markov Perfect Equilibrium," *Journal of Economic Theory*, 2001, *100* (2), 191–219.

Mastrobuoni, Giovanni and Matthew Weinberg, "Heterogeneity in Intra-Monthly Consumption Patterns, Self-Control, and Savings at Retirement," *American Economic Journal: Economic Policy*, 2009, *1* (2), 163–89.

McClure, Samuel M., Keith M. Ericson, David I. Laibson, George Loewenstein, and Jonathan D. Cohen, "Time Discounting for Primary Rewards," *Journal of Neuroscience*, 2007, *27* (21), 5796–5804.

Meier, Stephan and Charles Sprenger, "Present-Biased Preferences and Credit Card Borrowing," *American Economic Journal: Applied Economics*, 2010, *2* (1), 193–210.

Mian, Atif R., Ludwig Straub, and Amir Sufi, "The Saving Glut of the Rich and the Rise in Household Debt," *NBER w26941*, 2020.

Moser, Christian and Pedro Olea de Souza e Silva, "Optimal Paternalistic Savings Policies," *SSRN 2959844*, 2019.

Mullainathan, Sendhil, Joshua Schwartzstein, and William J. Congdon, "A Reduced-Form Approach to Behavioral Public Finance," *Annual Review of Economics*, 2011.

Nardi, Mariacristina De and Giulio Fella, "Saving and Wealth Inequality," *Review of Economic Dynamics*, 2017, *26*, 280–300.

O'Donoghue, Ted and Matthew Rabin, "Doing It Now or Later," *American Economic Review*, 1999, *89* (1), 103–124.

_ **and** _ , "Choice and Procrastination," *The Quarterly Journal of Economics*, 2001, *116* (1), 121–160.

_ **and** _ , "Optimal Sin Taxes," *Journal of Public Economics*, 2006, *90* (10-11), 1825–1849.

Olafsson, Arna and Michaela Pagel, "The Liquid Hand-To-Mouth: Evidence from Personal Finance Management Software," *The Review of Financial Studies*, 2018, *31* (11), 4398–4446.

Persson, Torsten and Lars E. O. Svensson, "Why a Stubborn Conservative Would Run a Deficit: Policy with Time-Inconsistent Preferences," *The Quarterly Journal of Economics*, 1989, *104* (2), 325–345.

Pollak, Robert A., "Consistent Planning," *The Review of Economic Studies*, 1968, *35* (2), 201–208.

Quadrini, Vincenzo, "The Importance of Entrepreneurship for Wealth Concentration and Mobility," *Review of Income and Wealth*, 1999, *45* (1), 1–19.

_ **and José-Víctor Ríos-Rull**, "Inequality in Macroeconomics," in "Handbook of Income Distribution," Vol. 2, Elsevier, 2015, pp. 1229–1302.

_ , _ **et al.**, "Understanding the U.S. Distribution of Wealth," *Federal Reserve Bank of Minneapolis Quarterly Review*, 1997, *21* (2), 22–36.

Skiba, Paige Marta and Jeremy Tobacman, "Payday Loans, Uncertainty, and Discounting: Explaining Patterns of Borrowing, Repayment, and Default," *Mimeo*, 2018.

Storesletten, Kjetil, Chris I. Telmer, and Amir Yaron, "The Welfare Cost of Business Cycles Revisited: Finite Lives and Cyclical Variation in Idiosyncratic Risk," *European Economic Review*, 2001, *45* (7), 1311–1339.

Strotz, Robert Henry, "Myopia and Inconsistency in Dynamic Utility Maximization," *The Review of Economic Studies*, 1956, *23* (3), 165–180.

Thaler, Richard, "Mental Accounting and Consumer Choice," *Marketing Science*, 1985, *4* (3), 199–214.

Tobacman, Jeremy, "The Partial Naivete Euler Equation," *Mimeo*, 2007.

Tourin, Agnes, "An Introduction to Finite Difference Methods for PDEs in Finance," *Optimal Stochastic Target Problems, and Backward SDE*, 2013.

Zeldes, Stephen P., "Optimal Consumption with Stochastic Income: Deviations from Certainty Equivalence," *The Quarterly Journal of Economics*, 1989, *104* (2), 275–298.

Zinman, Jonathan, "Household Debt: Facts, Puzzles, Theories, and Policies," *Annual Review of Economics*, 2015, *7* (1), 251–276.

APPENDIX

A Numerical Methods Theory

This paper uses finite difference methods to solve for the equilibrium of the IG agent (equations (7) – (8)). This section follows Achdou et al. (2020), who provide an excellent set of resources on these methods. I assume knowledge of finite difference methods here.

Barles and Souganidis (1991) show that a finite difference scheme converges to the unique viscosity solution of an HJB equation as long as certain conditions hold. As detailed below, these conditions do not necessarily hold when $\beta < 1$. This failure means that one cannot directly solve the Bellman of the IG agent. The key algorithmic insight of this paper is that the following two-step approach can be used to solve for the IG agent's equilibrium. First, solve the HJB equation of the dynamically consistent $\hat{u}$ agent. Second, compute the IG agent's equilibrium directly from the $\hat{u}$ agent using Propositions 10 and 11.

Failure of Monotonicity. Here I present a brief description of the problem: the Bellman equation of the IG agent fails to satisfy a monotonicity property. I follow Tourin (2013)'s treatment of Barles and Souganidis (1991). For simplicity, I assume here that income is deterministic, with $y_t = y$.

Let $\mathcal{G}$ denote the discretized grid over wealth a on which $v(a)$ is solved numerically. Assume this grid is uniformly spaced, and let Δ_a denote the size of the grid increment. At each gridpoint $g \in \mathcal{G}$, define:

$$S_g = \gamma v_g - u(c_g) - \frac{v_{g+1} - v_g}{\Delta_a}(y + ra_g - c_g)^+ - \frac{v_g - v_{g-1}}{\Delta_a}(y + ra_g - c_g)^-.$$

v_g, v_{g+1}, and v_{g-1} represent the value function at gridpoints $g, g+1$, and $g-1$, respectively. a_g is the wealth value at gridpoint g, and c_g is the consumption choice at gridpoint g.

For monotoncity to hold, S_g must be weakly decreasing in v_g, v_{g+1}, and v_{g-1}. To show that monotonicity fails when $\beta < 1$, assume that $y + ra_g - c_g < 0$. In this case, c_g is defined

implicitly by $u'(c_g) = \beta \frac{v_g - v_{g-1}}{\Delta_a}$. Consider an increase in v_{g-1}:

$$
\frac{\partial S_g}{\partial v_{g-1}} = -u'(c_g)\frac{\partial c_g}{\partial v_{g-1}} + \frac{1}{\Delta_a}(y + ra_g - c_g)^- + \frac{v_g - v_{g-1}}{\Delta_a}\frac{\partial c_g}{\partial v_{g-1}}
$$

$$
= (1 - \beta)\frac{v_g - v_{g-1}}{\Delta_a}\frac{\partial c_g}{\partial v_{g-1}} + \frac{1}{\Delta_a}(y + ra_g - c_g)^-,
$$

where the last line uses the property that $u'(c_g) = \beta \frac{v_g - v_{g-1}}{\Delta_a}$.

If $\beta = 1$ then monotonicity holds: $\frac{\partial S_g}{\partial v_{g-1}} < 0$ since the first term drops out, and $y + ra_g - c_g < 0$ by assumption.

If $\beta < 1$ then monotonicity may not hold. Since $\frac{\partial c_g}{\partial v_{g-1}} > 0$ the term $(1-\beta)\frac{v_g - v_{g-1}}{\Delta_a}\frac{\partial c_g}{\partial v_{g-1}} > 0$ whenever $\beta < 1$. Now, it is possible for $\frac{\partial S_g}{\partial v_{g-1}} > 0$, in which case monotonicity does not hold.

The above algebra points to the difficulty of using finite difference methods to solve for the Bellman of the IG agent. Since this difficulty only arises when $\beta < 1$, finite difference methods can still be used to solve for the equilibrium of the $\hat{u}$ agent. Given a solution to the $\hat{u}$ agent, the equilibrium of the IG agent can then be backed out: Proposition 10 implies that $v_j(a) = \hat{v}_j(a)$, and Proposition 11 defines $c_j(a)$ given $\hat{c}_j(a)$.

B   Present Bias and Policy: A Simple Example

This section provides a simple example to show how government intervention can improve the equilibrium of an economy with present-biased agents.[48] I study a simple "Eat-the-Pie" model of consumption-saving behavior. I assume that there is a single representative agent with initial wealth a_0. The model is deterministic, with $y_t \equiv \bar{y}$. The borrowing limit is set to the natural borrowing constraint of $\underline{a} = \frac{-\bar{y}}{r}$.

In this simple model, Proposition 5 implies that the IG agent consumes $c(a) = \frac{\rho-(1-\gamma)r}{\gamma-(1-\beta)}(a+\frac{\bar{y}}{r})$. However, the first-best consumption level is $\check{c}(a) = \frac{\rho-(1-\gamma)r}{\gamma}(a + \frac{\bar{y}}{r})$.[49]

For simplicity, I assume that $\rho = r$, $\gamma = 1$, and $a_0 > 0$. With these three assumptions, the first-best consumption level is $\check{c}(a_0) = ra_0 + \bar{y}$ and the first-best savings level is $\check{s}(a_0) = 0$. In other words, it is optimal for the agent to consume the annuity value of their wealth plus their deterministic income flow.

I now introduce a social planner to improve the consumption-saving decisions of the representative IG agent. This social planner is allowed to use a combination of interest rate subsidies and consumption taxes, subject to a balanced-budget constraint. Interest rate subsidies encourage saving, while consumption taxes are a means of financing these subsidies.

Denote the consumption tax by ϕ_t, and the subsidized interest rate by r_t^s. The social planner runs a balanced budget for all t, so the interest rate subsidy of $(r_t^s - r)a_t$ must equal the total tax revenue collected at each point in time.

With the introduction of consumption taxes, I will now use c_t to denote *gross* consumption expenditures at time t. However, the agent only gets to consume share $1 - \phi_t$ of gross expenditures, with the rest going to taxes.

The social planner can recover the first-best equilibrium using a constant consumption tax and interest rate subsidy. To implement the first-best equilibrium, the planner needs to

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[48]I thank David Laibson for suggesting this example. A similar result is presented in Laibson (1998).

[49]As discussed in Section 5.4, IG preferences feature a single welfare criterion even though they are time-inconsistent.

choose r^s and ϕ such that:

$$(1 - \phi)\frac{ra_0 + \frac{r\bar{y}}{r^s}}{\beta} = ra_0 + \bar{y} \tag{36}$$

$$\phi\frac{ra_0 + \frac{r\bar{y}}{r^s}}{\beta} = (r^s - r)a_0 \tag{37}$$

Under the simple calibration studied here, the IG agent will choose a gross consumption expenditure of $c(a) = \frac{ra + \frac{r\bar{y}}{r^s}}{\beta}$.[50] However, actual consumption is only $(1 - \phi)c(a)$. Equation (36) imposes that realized consumption is at its first-best level: $(1 - \phi)c(a_0) = ra_0 + \bar{y}$. Equation (37) is the balanced-budget condition. It says that tax revenues of $\phi c(a_0)$ must equal the interest rate subsidy of $(r^s - r)a_0$.

One can show that the following set of policy tools produces the first-best equilibrium:

$$r^s = \frac{r}{\beta}$$

$$\phi = \frac{ra_0(1 - \beta)}{ra_0 + \beta\bar{y}}$$

For example, consider the calibration $\beta = 0.75$, $r = 3\%$, $\bar{y} = 1$, and $a_0 = 3$. The optimal consumption tax is $\phi = 2.67\%$ and the optimal subsidized interest rate is $r^s = 4\%$.

Welfare and Implementability when $\beta = 1$. Proposition 21 highlights the channels through which present bias can matter for policymakers: present bias does not matter for determining whether a policy is welfare-improving, but does matter for determining whether a policy is feasible. This toy model can be used to formalize this discussion.

Proposition 21 implies that this interest rate subsidy plus consumption tax policy will be welfare-improving for the $\beta = 1$ agent. However, this policy is not implementable in an economy populated by a representative $\beta = 1$ agent. At time 0, the $\beta = 1$ agent will consume $r(a_0 + \frac{\bar{y}}{r^s})$, which will be too low to generate the requisite taxes needed to support the interest rate subsidy of $(r^s - r)a_0$.

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[50]This consumption function uses the property that a constant consumption tax does not change the gross expenditure of the IG agent. This property is derived formally in the proof of Proposition 20.

C Proofs

Throughout this appendix, it is assumed that the reader understands the construction of the $\hat{u}$ agent. Details of the $\hat{u}$ construction are given in Section 4, and in Harris and Laibson (2013).

C.1 Proof of Proposition 1

The proof of Proposition 1 relies on Proposition 10.

Lemma 22. *The value function of the $\hat{u}$ agent, denoted $\hat{v}_j(a)$, is unique.*

Proof. See Harris and Laibson (2013) for full details. Intuitively, the $\hat{u}$ agent is an exponential discounter who optimally choses consumption to maximize $\hat{v}_j(a)$. Since there is only one maximal value function, $\hat{v}_j(a)$ must be unique. □

The proof of Proposition 1 follows immediately from Lemma 22 and Proposition 10. In particular, Lemma 22 says $\hat{v}$ is unique, and Proposition 10 says $v = \hat{v}$. Hence, v is unique.

C.2 Proof of Corollary 3

This proof extends the $\beta = 1$ case of Achdou et al. (2020). A similar result is given in Harris and Laibson (2004). Taking a derivative of (7) with respect to a and applying the first-order condition (8) gives

$$\left[(\rho - r) + (1 - \beta)c_j'(a)\right] u'(c_j(a)) = u''(c_j(a))c_j'(a)s_j(a) + \lambda_j(u'(c_{-j}(a)) - u'(c_j(a))).$$

Applying Ito's Lemma to $u'(c_j(a_t))$ gives $\mathbb{E}_t du'(c_j(a_t)) = u''(c_j(a_t))c_j'(a_t)da_t + \lambda_j(u'(c_{-j}(a)) - u'(c_j(a)))dt$. Since $da_t = s_j(a_t)dt$, we have $\mathbb{E}_t du'(c_j(a_t)) = u''(c_j(a))c_j'(a)s_j(a)dt + \lambda_j(u'(c_{-j}(a)) - u'(c_j(a)))dt$. Rearranging gives (13).

C.3 Proof of Proposition 4

For full details, see Theorem 21 of Harris and Laibson (2004). The value function for the IG agent is given by (see equation (7)):

$$\rho v_j(a) = u(c_j(a)) + v'_j(a)(y_j + ra - c_j(a)) + \lambda_j(v_{-j}(a) - v_j(a)).$$

If the constraint binds at $\underline{a}$ for income state j, then $c_j(\underline{a}) = y_j + r\underline{a}$. Thus,

$$\rho v_j(\underline{a}) = u(y_j + r\underline{a}) + \lambda_j(v_{-j}(\underline{a}) - v_j(\underline{a})).$$

Since the value function is continuous, $\rho v_j(\underline{a}) = \lim_{a \to {}^+\underline{a}} \rho v_j(a)$. Therefore:

$$u(y_j + r\underline{a}) + \lambda_j(v_{-j}(\underline{a}) - v_j(\underline{a})) = \lim_{a \to {}^+\underline{a}} \left[u(c_j(a)) + v'_j(a)(y_j + ra - c_j(a)) + \lambda_j(v_{-j}(a) - v_j(a)) \right],$$

or simply

$$u(y_j + r\underline{a}) = \lim_{a \to {}^+\underline{a}} \left[u(c_j(a)) + v'_j(a)(y_j + ra - c_j(a)) \right].$$

Using equation (8) gives

$$u(y_j + r\underline{a}) = \lim_{a \to {}^+\underline{a}} \left[u(c_j(a)) + \frac{1}{\beta} u'(c_j(a))(y_j + ra - c_j(a)) \right].$$

This equation can be rearranged to yield:

$$\lim_{a \to {}^+\underline{a}} u'(c_j(a)) = \beta \frac{\lim_{a \to {}^+\underline{a}} u(c_j(a)) - u(y_j + r\underline{a})}{\lim_{a \to {}^+\underline{a}} c_j(a) - (y_j + r\underline{a})}.$$

C.4 Proof of Proposition 5

The proof of Achdou et al. (2020)'s Proposition 2 applies to the $\hat{u}$ agent, giving

$$\lim_{a \to \infty} \hat{c}_j(a) = \frac{\rho - (1 - \gamma)r}{\gamma} a.$$

Since the IG agent sets $c_j(a) = \frac{1}{\psi}\hat{c}_j(a)$ (see Proposition 11), this gives

$$\lim_{a \to \infty} c_j(a) = \frac{1}{\psi} \frac{\rho - (1-\gamma)r}{\gamma}$$
$$= \frac{\rho - (1-\gamma)r}{\gamma - (1-\beta)} a$$

The proof for $\lim_{s \to \infty} s_j(a)$ is similar.

C.5 Proof of Corollary 6

Using Itô's Lemma, $\mathbb{E}_t \frac{dc_j(a_t)/dt}{c_j(a_t)} = \frac{c'_j(a)s_j(a) + \lambda_j(c_{-j}(a_t) - c_j(a_t))}{c_j(a)}$. Equations (15) and (16) give $\lim_{a \to \infty} \mathbb{E}_t \frac{dc_j(a_t)/dt}{c_j(a_t)} = \frac{r\beta - \rho}{\gamma - (1-\beta)}$. Taking a derivative with respect to r completes the proof.

C.6 Proof of Proposition 10

In this section I prove value function equivalence between the IG agent and the $\hat{u}$ agent. This proof is similar to Harris and Laibson (2013) and Laibson and Maxted (2020), and is included for complete detail.

Remark 23. *Most of the complexity in this proof arises in allowing for $\underline{a}$ to bind. The proof simplifies when $\underline{a}$ is the natural borrowing limit.*

The $\hat{u}$ Agent's HJBVI. To begin, I express the $\hat{u}$ agent's value function recursively. In the model of Section 3.1, $\hat{v}_j(a)$ is a viscosity solution to the following Hamilton-Jacobi-Bellman Variational Inequality (HJBVI):[51]

$$0 = \min \left\{ \rho \hat{v}_j(a) - \max_{\hat{c}} \hat{u}^+(\hat{c}) + \hat{v}'_j(a)(y_j + ra - \hat{c}) + \lambda_j(\hat{v}_{-j}(a) - \hat{v}_j(a)), \ \hat{v}_j(a) - \hat{v}_j^* \right\}, \quad (38)$$

---

[51] For details on HJBVIs, and for additional economic applications, see the Stopping Time Problems at https://benjaminmoll.com/codes/.

where $\hat{v}_j^* = \frac{u(y_j + r\underline{a}) + \lambda_j \hat{v}_{-j}(\underline{a})}{\rho + \lambda_j}$.[52] Additionally, HJBVI equation (38) is subject to the following boundary condition at $\underline{a}$:

$$0 \leq \left[\hat{v}_j'(\underline{a}) - \frac{\partial \hat{u}^+(\psi(y_j + r\underline{a}))}{\partial \hat{c}} \right] \left(\hat{v}_j(\underline{a}) - \hat{v}_j^* \right). \tag{39}$$

This boundary condition is a shorthand way to restrict the consumption decision of the $\hat{u}$ agent at the borrowing limit $\underline{a}$. In particular, it ensures that the $\hat{u}$ agent either sets $\hat{c}_j(\underline{a}) \leq \psi(y_j + r\underline{a})$ or else sets $\hat{c}_j(\underline{a}) = y_j + r\underline{a}$ (details below).

The "variational inequality" structure of equation (38) captures the option value that is inherent in the $\hat{u}$ utility function. The left branch of equation (38) is the standard HJB of an exponential discounter with utility function $\hat{u}^+$. The right branch of equation (38) captures the agent's ability to "stop." Stopping yields the value $\hat{v}_j^* = \frac{u(y_j + r\underline{a}) + \lambda_j \hat{v}_{-j}(\underline{a})}{\rho + \lambda_j}$. This stopping condition can be thought of as giving the $\hat{u}$ agent the option to trade away all wealth above $\underline{a}$ in exchange for a utility flow of $u(y_j + r\underline{a})$ that persists until the agent's income state changes.[53]

For $a > \underline{a}$ it will always be optimal to select the left branch of equation (38) (i.e., do not "stop"). When $a > \underline{a}$ the $\hat{u}$ agent can always choose $\hat{c}$ such that $\hat{u}^+(\hat{c}) = u(y_j + r\underline{a})$, and this utility flow is attained without trading away all wealth above $\underline{a}$.

For $a = \underline{a}$, the $\hat{u}$ agent faces a choice. If the agent chooses the left branch of equation (38) then the lower utility function $\hat{u}^+$ is obtained. But, the benefit of choosing the left branch is that $\hat{c}_j(\underline{a}) \leq \psi(y_j + r\underline{a})$ and therefore wealth is accumulated since $\hat{s}_j(\underline{a}) > 0$. If the agent chooses the right branch of equation (38) at $a = \underline{a}$ then the agent earns a larger utility flow of $u(y_j + r\underline{a})$, but remains stuck at wealth level $\underline{a}$. Thus, equation (38) captures the tradeoff that the $\hat{u}$ agent faces at $a = \underline{a}$.

The boundary condition in equation (39) is a way to restrict the behavior of the $\hat{u}$ agent at $\underline{a}$. The $\hat{u}$ agent must choose either to "continue" by setting $\hat{c}_j(\underline{a}) \leq \psi(y_j + r\underline{a})$, or else to "stop." Stopping implies $\hat{v}_j(\underline{a}) = \hat{v}_j^*$, in which case equation (39) holds. If the agent chooses to continue, meaning that $\hat{v}_j(\underline{a}) \geq \hat{v}_j^*$, boundary condition (39) imposes that

---

[52]This implicitly assumes that $\underline{a} > \frac{-y_1}{r}$. If $\underline{a} = \frac{-y_1}{r}$ then $\hat{v}_j^* = -\infty$. In this case, $\hat{v}_j^*$ is never chosen.

[53]If the agent chooses this alternate "stopping" option, their value function is given by $\rho \hat{v}_j(\underline{a}) = u(y_j + r\underline{a}) + \lambda_j(\hat{v}_{-j}(\underline{a}) - \hat{v}_j(\underline{a}))$. Solving for $\hat{v}_j(\underline{a})$ gives the formula for $\hat{v}_j^*$.

$\hat{v}'_j(\underline{a}) \geq \frac{\partial \hat{u}^+(\psi(y_j + r\underline{a}))}{\partial \hat{c}}$. Since the $\hat{u}$ agent sets $\frac{\partial \hat{u}^+(\hat{c}_j(\underline{a}))}{\partial \hat{c}} = \hat{v}'_j(\underline{a})$ in the continuation region, this lower bound on $\hat{v}'_j(\underline{a})$ ensures that $\hat{c}_j(\underline{a}) \leq \psi(y_j + r\underline{a})$.

Proof Intuition. The intuition for this proof is as follows. Assume that $v_j(a) = \hat{v}_j(a)$ and $a > \underline{a}$. Then, differential equations (7) and (38) can be combined to yield:

$$u(c_j(a)) - v'_j(a)c_j(a) = \hat{u}^+(\hat{c}_j(a)) - \hat{v}'_j(a)\hat{c}_j(a).$$

Utility function $\hat{u}$ is reverse-engineered so that this condition holds.

The IG Agent: A Modified Bellman Equation. Following Theorem 2 of Harris and Laibson (2013), let $f^+(\alpha)$ be the unique value of c satisfying $u'(c) = \alpha$. Let $h^+(\alpha) = u(f^+(\beta\alpha)) - \alpha f^+(\beta\alpha)$. Since $u'(c_j(a)) = \beta v'_j(a)$ for $a > \underline{a}$, it is the case that $h^+(v'_j(a)) = u(f^+(\beta v'_j(a))) - v'_j(a)f^+(\beta v'_j(a)) = u(c_j(a)) - v'_j(a)c_j(a)$.

Next, let $\underline{f}_j(\alpha)$ be the unique value of c satisfying $u'(c) = \max\{u'(y_j + r\underline{a}), \alpha\}$. Let $\underline{h}_j(\alpha) = u(\underline{f}_j(\beta\alpha)) - \alpha\underline{f}_j(\beta\alpha)$.

Define:

$$h_j(\alpha, a) = \begin{cases} h^+(\alpha) & \text{if} \quad a > \underline{a} \\ \underline{h}_j(\alpha) & \text{if} \quad a = \underline{a} \end{cases}.$$

Function h can be used to rewrite equation the Bellman equation of the IG agent (equations (7) and (8)) as follows:

$$\rho v_j(a) = v'_j(a)(y_j + ra) + \lambda_j(v_{-j}(a) - v_j(a)) + h_j(v'_j(a), a). \tag{40}$$

The $\hat{u}$ Agent: A Modified Bellman Equation. At $a = \underline{a}$, the $\hat{u}$ agent faces a choice to "continue" or to "stop". Continuing attains utility function $\hat{u}^+$ with consumption $\hat{c}_j(\underline{a}) \leq \psi(y_j + r\underline{a})$. Stopping attains utility function u with consumption $\hat{c}_j(\underline{a}) = y_j + r\underline{a}$.

Lemma 24. *The $\hat{u}$ agent will choose to continue at $\underline{a}$ when $\hat{v}'_j(\underline{a}) > \frac{1}{\beta}(y_j + r\underline{a})^{-\gamma}$. The $\hat{u}$ agent will choose to stop when $\hat{v}'_j(\underline{a}) < \frac{1}{\beta}(y_j + r\underline{a})^{-\gamma}$.*

Proof. If the $\hat{u}$ agent chooses to continue at $\underline{a}$ then the $\hat{u}$ agent sets $\frac{\partial \hat{u}^+(\hat{c}_j(\underline{a}))}{\partial \hat{c}} = \hat{v}'_j(\underline{a})$. This

implies $\hat{c}_j(\underline{a}) = \psi(\beta \hat{v}'_j(\underline{a}))^{-\frac{1}{\gamma}}$. This consumption choice yields a value at $\underline{a}$ of

$$\rho \hat{v}_j(\underline{a}) = \hat{u}^+(\hat{c}_j(\underline{a})) + \frac{\psi^\gamma}{\beta} \hat{c}_j(\underline{a})^{-\gamma}(y_j + r\underline{a} - \hat{c}_j(\underline{a})) + \lambda_j(\hat{v}_{-j}(\underline{a}) - \hat{v}_j(\underline{a})). \tag{41}$$

If the $\hat{u}$ agent chooses to stop, the value at $\underline{a}$ is given by

$$\rho \hat{v}_j(\underline{a}) = u(y_j + r\underline{a}) + \lambda_j(\hat{v}_{-j}(\underline{a}) - \hat{v}_j(\underline{a})). \tag{42}$$

Comparing (41) and (42), one can show that the $\hat{u}$ agent is indifferent between the two choices when $\hat{v}'_j(\underline{a}) = \frac{1}{\beta}(y_j + r\underline{a})^{-\gamma}$ (which implies $\hat{c}_j(\underline{a}) = \psi(y_j + r\underline{a})$). The $\hat{u}$ agent chooses value function (41) when $\hat{v}'_j(\underline{a}) > \frac{1}{\beta}(y_j + r\underline{a})^{-\gamma}$, and value function (42) when $\hat{v}'_j(\underline{a}) < \frac{1}{\beta}(y_j + r\underline{a})^{-\gamma}$. $\qquad\square$

Using Lemma 24, I now proceed as in the IG case. For the $\hat{u}$-agent, let $\hat{f}^+(\alpha)$ be the unique value of $\hat{c}$ satisfying $\frac{\partial \hat{u}^+(\hat{c})}{\hat{c}} = \alpha$. Let $\hat{h}^+(\alpha) = \hat{u}^+(\hat{f}^+(\alpha)) - \alpha \hat{f}^+(\alpha)$. Since the $\hat{u}$ agent sets $\frac{\partial \hat{u}^+(\hat{c}_j(a))}{\partial \hat{c}} = \hat{v}'_j(a)$ for $a > \underline{a}$, it is the case that $\hat{h}^+(\hat{v}'_j(a)) = \hat{u}^+(\hat{f}^+(\hat{v}'_j(a))) - \hat{v}'_j(a)\hat{f}^+(\hat{v}'_j(a)) = \hat{u}^+_j(\hat{c}_j(a)) - \hat{v}'_j(a)\hat{c}_j(a)$.

Next, let $\underline{\hat{f}}_j(\alpha)$ be the unique value of $\hat{c}$ satisfying:

$$\hat{c} = \begin{cases} \psi(\beta\alpha)^{-\frac{1}{\gamma}} & \text{if} \quad \alpha \geq \frac{1}{\beta}(y_j + r\underline{a})^{-\gamma} \\ y_j + r\underline{a} & \text{otherwise} \end{cases}.$$

Let $\underline{\hat{h}}_j(\alpha) = \hat{u}_j(\underline{\hat{f}}_j(\alpha), \underline{a}) - \alpha\underline{\hat{f}}_j(\alpha)$.

Define:

$$\hat{h}_j(\alpha, a) = \begin{cases} \hat{h}^+(\alpha) & \text{if} \quad a > \underline{a} \\ \underline{\hat{h}}_j(\alpha) & \text{if} \quad a = \underline{a} \end{cases}.$$

Using Lemma 24, function $\hat{h}$ can be used to rewrite the Bellman equation of the $\hat{u}$ agent (equation (38)) as follows:

$$\rho \hat{v}_j(a) = \hat{v}'_j(a)(y_j + ra) + \lambda_j(\hat{v}_{-j}(a) - \hat{v}_j(a)) + \hat{h}_j(\hat{v}'_j(a), a) \tag{43}$$

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From inspection, we can see that equations (40) and (43) are identical if and only if h_j and $\hat{h}_j$ are the same. This can be confirmed directly.

C.7 Proof of Proposition 11

For $a > \underline{a}$, equation (8) specifies that the IG agent sets $u'(c_j(a)) = \beta v'_j(a)$. The $\hat{u}$ agent sets $\frac{\partial \hat{u}^+(\hat{c}_j(a))}{\partial \hat{c}} = \hat{v}'_j(a)$. Imposing the value function equivalence that $v_j(a) = \hat{v}_j(a)$ (Proposition 10) gives:

$$\frac{\partial u(c_j(a))}{\partial c} = \beta \frac{\partial \hat{u}^+(\hat{c}_j(a))}{\partial \hat{c}}.$$

This implies

$$c_j(a) = \frac{1}{\psi}\hat{c}_j(a).$$

For $a = \underline{a}$, consider first the case where $\hat{c}_j(\underline{a}) \leq \psi(y_j + r\underline{a})$. If the $\hat{u}$ agent sets $\hat{c}_j(\underline{a}) \leq \psi(y_j + r\underline{a})$ then this implies that $\hat{v}'_j(\underline{a}) \geq \frac{\partial \hat{u}^+(\psi(y_j+r\underline{a}))}{\partial \hat{c}} = \frac{1}{\beta}(y_j + r\underline{a})^{-\gamma}$. Since $v_j(a) = \hat{v}_j(a)$ by value function equivalence (Proposition 10), this also means that $\beta v'_j(\underline{a}) \geq (y_j + r\underline{a})^{-\gamma}$. In this case, equation (8) specifies that the IG agent sets $u'(c_j(\underline{a})) = \beta v'_j(\underline{a})$. The argument that was just used for $a > \underline{a}$ continues to hold here.

Next, consider the case where $\hat{c}_j(\underline{a}) = y_j + r\underline{a}$. If the $\hat{u}$ agent sets $\hat{c}_j(\underline{a}) = y_j + r\underline{a}$ then it must be that $\hat{v}'_j(\underline{a}) \leq \frac{1}{\beta}(y_j + r\underline{a})^{-\gamma}$ (Lemma 24). By optimality condition (8), the IG agent also sets $c_j(\underline{a}) = y_j + r\underline{a}$.

C.8 Proof of Proposition 13

This proof uses the property that the $\hat{u}$ agent behaves identically to a standard exponential agent when $\underline{a}$ is the natural borrowing limit (Remark 12).

First consider the case where $r \leq \rho$. The Euler equation of the standard exponential agent implies that $\check{c}(a) \geq y + ra$ for all $a \geq 0$ (see Achdou et al. (2020)). Since the IG agent sets $c(a) = \frac{1}{\psi}\hat{c}(a) = \frac{1}{\psi}\check{c}(a)$, the IG agent strictly dissaves for all $a \geq 0$ when $r \leq \rho$.

Next consider the case where $r \in (\rho, \frac{\rho}{\beta})$. The standard exponential agent consumes

according to $\check{c}(a) = \frac{\rho-(1-\gamma)r}{\gamma}(a + \frac{y}{r})$ for $a \geq 0$ (see e.g. Fagereng et al. (2019) for a proof). The IG agent therefore sets $c(a) = \frac{1}{\psi}\check{c}(a) = \frac{\rho-(1-\gamma)r}{\gamma-(1-\beta)}(a + \frac{y}{r})$ for $a \geq 0$. One can show that $s(a) = y + ra - c(a) < 0$ whenever $r < \frac{\rho}{\beta}$. Thus, the IG agent strictly dissaves for all $a \geq 0$ when $r \in (\rho, \frac{\rho}{\beta})$.

In both cases the IG agent strictly dissaves for all $a \geq 0$. This means that the IG agent dissaves at $a = 0$, completing the proof that $s(0) < 0$ whenever $r < \frac{\rho}{\beta}$. This holds regardless of how large $\Gamma(0)$ is.

Note that this proof does not rely on some sort of consumption discontinuity at $a = 0$. The consumption function $c(a)$ is continuous at $a = 0$. To show this, recall that the IG agent's value function is given by

$$\rho v(a) = u(c(a)) + v'(a)(\varsigma(a)a + y - c(a)).$$

The IG agent sets $u'(c(a)) = \beta v'(a)$. Therefore

$$\rho v(a) = u(c(a)) + \frac{c(a)^{-\gamma}}{\beta}(\varsigma(a)a + y - c(a)).$$

Since $v(a)$ is continuous and $\varsigma(a)a$ is continuous, $c(a)$ is also continuous at $a = 0$.

C.9 Proof of Proposition 14

Value Function Uniqueness. I first prove that the IG agent's intrapersonal game features a unique value function.

Lemma 25. *For the two-asset model described in Section 5.2, the value function of the $\hat{u}$ agent, denoted $\hat{v}_j(a, \zeta)$, is equivalent to the value function $v_j(a, \zeta)$ of the IG agent.*

Proof. In this extended model, $\hat{v}_j(a, \zeta)$ is a viscosity solution to the following Hamilton-Jacobi-Bellman Variational Inequality (HJBVI):

$$0 = \min\left\{\hat{v}_j(a, \zeta) - \hat{v}_j^*(\zeta), \ \rho\hat{v}_j(a, \zeta) - \max_{\hat{c},\hat{d}} \ \hat{u}^+(\hat{c}) + \frac{\partial\hat{v}_j(a, \zeta)}{\partial a}(y_j + \varsigma(a)a - \hat{d} - \chi(\hat{d}, \zeta) - \hat{c}) \right.$$
$$\left. + \frac{\partial\hat{v}_j(a, \zeta)}{\partial\zeta}(\zeta r^\varsigma + \hat{d}) + \lambda_j(\hat{v}_{-j}(a, \zeta) - \hat{v}_j(a, \zeta)) + \frac{1}{2}\frac{\partial^2\hat{v}_j(a, \zeta)}{\partial\zeta^2}(\zeta\sigma^\varsigma)^2\right\}, \tag{44}$$

where $\hat{v}_j^*(\zeta) = \frac{u(y_j + r\underline{a}) + \lambda_j \hat{v}_{-j}(\underline{a}, \zeta)}{\rho + \lambda_j}$.[54] HJBVI equation (44) is subject to the following boundary condition at $\underline{a}$:

$$0 \leq \left[\hat{v}_j'(\underline{a}, \zeta) - \frac{\partial \hat{u}^+(\psi(y_j + r\underline{a}))}{\partial \hat{c}} \right] \left[\hat{v}_j(\underline{a}, \zeta) - \hat{v}_j^*(\zeta) \right]. \tag{45}$$

Given this setup, the proof of value function equivalence is the same as Proposition 10. Asset allocation choice $d_j(a, \zeta)$ adds no additional difficulty because the $\hat{u}$ agent and the IG agent both utilize the same first-order condition to choose $d_j(a, \zeta)$. $\qquad\square$

Given Lemma 25, the proof of value function uniqueness is the same as Proposition 1.

Independence of Asset Allocation Decision and β. When $\underline{a}$ is the natural borrowing limit, the proof that $d_j(a, \zeta)$ is independent of β is as follows. By Lemma 25, $v_j(a, \zeta) = \hat{v}_j(a, \zeta)$. Given value function equivalence, equation (27) implies that the IG agent chooses the same illiquid asset policy function as the $\hat{u}$ agent: $d_j(a, \zeta) = \hat{d}_j(a, \zeta)$. When $\underline{a}$ is the natural borrowing limit the $\hat{u}$ agent behaves identically to a standard exponential agent (Remark 12). Thus, $d_j(a, \zeta) = \hat{d}_j(a, \zeta) = \check{d}_j(a, \zeta)$.

C.10 Proof of Proposition 16

The (potentially naive) IG agent sets $u'(c_j(a, \zeta)) = \beta \frac{\partial v_j^E(a, \zeta)}{\partial a}$. By Lemma 25, one can construct a $\hat{u}$ agent using β^E such that $\hat{v}_j(a, \zeta) = v_j^E(a, \zeta)$. This $\hat{u}$ agent chooses consumption such that $\frac{\partial \hat{u}^+(\hat{c}_j(a, \zeta))}{\partial \hat{c}} = \frac{\partial \hat{v}_j(a, \zeta)}{\partial a}$. This implies that $\frac{\partial \hat{v}_j(a, \zeta)}{\partial a} = \frac{(\psi^E)^\gamma}{\beta^E} \hat{c}_j(a, \zeta)^{-\gamma}$, where $\psi^E = \frac{\gamma - (1 - \beta^E)}{\gamma}$.

Using the value function equivalence property that $\hat{v}_j(a, \zeta) = v_j^E(a, \zeta)$:

$$u'(c_j(a, \zeta)) = \beta \frac{(\psi^E)^\gamma}{\beta^E} \hat{c}_j(a, \zeta)^{-\gamma}.$$

Rearranging gives

$$c_j(a, \zeta) = \left(\frac{\beta^E}{\beta} \right)^{\frac{1}{\gamma}} \frac{1}{\psi^E} \times \hat{c}_j(a, \zeta).$$

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[54] This implicitly assumes that $\underline{a} > \frac{-y_1}{r}$. If $\underline{a} = \frac{-y_1}{r}$ then $\hat{v}_j^*(\underline{a}) = -\infty$. In this case, $\hat{v}_j^*$ is never chosen.

To complete the proof, note that the $\hat{u}$ agent behaves identically to a standard exponential agent when $\underline{a}$ does not bind (Remark 12). This implies that the $\hat{u}$ agent sets $\hat{c}_j(a, \zeta) = \check{c}_j(a, \zeta)$ regardless of β and β^E. Therefore $c_j(a, \zeta) = \left(\frac{\beta^E}{\beta}\right)^{\frac{1}{\gamma}} \frac{1}{\psi^E} \times \check{c}_j(a, \zeta)$, as desired.

To see when consumption is increasing in naivete, consider:

$$\frac{\partial c_j(a, \zeta)}{\partial \beta^E} \propto \frac{1}{\gamma} \left(\frac{\beta^E}{\beta}\right)^{\frac{1-\gamma}{\gamma}} \frac{1}{\beta} \frac{1}{\psi^E} - \left(\frac{\beta^E}{\beta}\right)^{\frac{1}{\gamma}} \frac{1}{\psi^E} \frac{1}{\gamma - (1 - \beta^E)}$$

$$\propto \frac{1}{\beta^E} - \frac{1}{\psi^E}$$

For $\beta^E < 1$, one can show that $\psi^E > \beta^E$ when $\gamma > 1$, and $\psi^E < \beta^E$ when $\gamma < 1$. Thus, consumption is increasing in naivete when $\gamma > 1$, and decreasing in naivete when $\gamma < 1$.

C.11   Proof of Corollary 19

In the model of Section 3, the expected continuation-value function $v_j^E(a)$ is characterized as follows:

$$\rho v_j^E(a) = u(c_j^E(a)) + \frac{\partial v_j^E(a)}{\partial a}(y_j + ra - c_j^E(a)) + \lambda_j(v_{-j}^E(a) - v_j^E(a)), \tag{46}$$

$$u'(c_j^E(a)) = \begin{cases} \beta^E \frac{\partial v_j^E(a)}{\partial a} & \text{if } a > \underline{a} \\ \max\{\beta^E \frac{\partial v_j^E(\underline{a})}{\partial a}, u'(y_j + r\underline{a})\} & \text{if } a = \underline{a} \end{cases}. \tag{47}$$

Equations $(46) - (47)$ are identical to $(7) - (8)$ except that the true short-run discount factor β is replaced by the perceived discount factor β^E. The agent's actual consumption decision is given by:

$$u'(c_j(a)) = \begin{cases} \beta \frac{\partial v_j^E(a)}{\partial a} & \text{if } a > \underline{a} \\ \max\{\beta \frac{\partial v_j^E(\underline{a})}{\partial a}, u'(y_j + r\underline{a})\} & \text{if } a = \underline{a} \end{cases}.$$

Let $s_j^E(a) = y_j + ra - c_j^E(a)$ denote the perceived savings rate. Taking a derivative of

(46) with respect to a and applying the first-order condition $u'(c_j^E(a)) = \beta^E \frac{\partial v_j^E(a)}{\partial a}$ gives

$$\left[(\rho - r) + \left(1 - \beta^E\right) \frac{\partial c_j^E(a)}{\partial a}\right] \frac{\partial v_j^E(a)}{\partial a} = \frac{\partial^2 v_j^E(a)}{\partial a^2} s_j^E(a) + \lambda_j \left(\frac{\partial v_{-j}^E(a)}{\partial a} - \frac{\partial v_j^E(a)}{\partial a}\right).$$

Multiplying through by β and using the property that $u'(c_j(a)) = \beta \frac{\partial v_j^E(a)}{\partial a}$ gives:

$$\left[(\rho - r) + \left(1 - \beta^E\right) \frac{\partial c_j^E(a)}{\partial a}\right] u'(c_j(a)) = u''(c_j(a)) \frac{\partial c_j(a)}{\partial a} s_j^E(a) + \lambda_j \left(u'(c_{-j}(a)) - u'(c_j(a))\right).$$

Applying Ito's Lemma gives $\mathbb{E}_t du'(c_j(a_t)) = u''(c_j(a_t)) c_j'(a_t) s_j(a_t) + \lambda_j (u'(c_{-j}(a)) - u'(c_j(a))) dt$:

$$\left[(\rho - r) + \left(1 - \beta^E\right) \frac{\partial c_j^E(a)}{\partial a}\right] u'(c_j(a)) = \mathbb{E}_t[du'(c_j(a_t))/dt] + u''(c_j(a)) \frac{\partial c_j(a)}{\partial a} (s_j^E(a) - s_j(a))$$

$$= \mathbb{E}_t[du'(c_j(a_t))/dt] + u''(c_j(a)) \frac{\partial c_j(a)}{\partial a} (c_j(a) - c_j^E(a)).$$

The first-order conditions of $u'(c_j^E(a)) = \beta^E \frac{\partial v_j^E(a)}{\partial a}$ and $u'(c_j(a)) = \beta \frac{\partial v_j^E(a)}{\partial a}$ imply that $c_j^E(a) = \left(\frac{\beta}{\beta^E}\right)^{\frac{1}{\gamma}} c_j(a)$. Thus,

$$\left[(\rho - r) + \left(1 - \beta^E\right) \left(\frac{\beta}{\beta^E}\right)^{\frac{1}{\gamma}} \frac{\partial c_j(a)}{\partial a}\right] u'(c_j(a)) = \mathbb{E}_t[du'(c_j(a_t))/dt] + u''(c_j(a)) \frac{\partial c_j(a)}{\partial a} c_j(a) \left(1 - \left(\frac{\beta}{\beta^E}\right)^{\frac{1}{\gamma}}\right)$$

Dividing through by marginal utility and using the property that $\gamma = \frac{-cu''(c)}{u'(c)}$:

$$\left[(\rho - r) + \left(1 - \beta^E\right) \left(\frac{\beta}{\beta^E}\right)^{\frac{1}{\gamma}} \frac{\partial c_j(a)}{\partial a}\right] = \frac{\mathbb{E}_t[du'(c_j(a_t))/dt]}{u'(c_j(a))} - \gamma \frac{\partial c_j(a)}{\partial a} \left(1 - \left(\frac{\beta}{\beta^E}\right)^{\frac{1}{\gamma}}\right).$$

Rearranging yields the desired result.

C.12 Proof of Proposition 20

Step 1: Value Function Equivalence for the Naive Agent $(\gamma \neq 1)$. Recall that the $\hat{u}$ utility function is constructed so that the value function of the sophisticated IG agent is equivalent to the value function of an exponential agent with utility function $\hat{u}$. The first

step of this proof generalizes this construction to allow for naivete. I construct a utility function, denoted $\hat{\hat{u}}$, such that the *realized* value function of the (potentially naive) IG is equivalent to the value function of an exponential agent with utility function $\hat{\hat{u}}$. I refer to this agent as the $\hat{\hat{u}}$ agent.

Note that when $\beta^E \neq \beta$ the realized value function of the naive IG agent does not equal the naif's expected value function. As given in the main text, the expected continuation-value function is $v_t^E = \mathbb{E}_t \left[\int_t^\infty e^{-\rho(s-t)} u(c_s^E) ds \right]$. However, the realized value function is

$$v_t^R = \mathbb{E}_t \left[\int_t^\infty e^{-\rho(s-t)} u(c_s) ds \right].$$

v^R is based on the naif's realized consumption, while v^E is based on their perceived consumption.

Let $\hat{\hat{u}}(c) = \frac{xc^{1-\gamma}-1}{1-\gamma}$. This is a positive affine transformation of CRRA utility function $u(c)$ whenever $x > 0$. When this is the case, the $\hat{\hat{u}}$ agent will behave identically to a standard exponential agent. Thus, I will use $\check{c}_j(a, \zeta)$ to refer to the consumption of the $\hat{\hat{u}}$ agent.

In order to generate value function equivalence between the (possibly naive) IG agent and the $\hat{\hat{u}}$ agent, I construct $\hat{\hat{u}}$ so that the following condition holds for all $a > \underline{a}$:

$$u(c_j(a,\zeta)) - \frac{\partial v_j^R(a,\zeta)}{\partial a} c_j(a,\zeta) = \hat{\hat{u}}(\check{c}_j(a,\zeta)) - \frac{\partial v_j^R(a,\zeta)}{\partial a} \check{c}_j(a,\zeta). \tag{48}$$

This condition ensures that $v_j^R(a, \zeta) = \hat{\hat{v}}_j(a, \zeta)$ whenever $\underline{a}$ does not bind in equilibrium. See the proof of Proposition 10 for details.

I want to solve for x such that equation (48) holds. From Proposition 16, note that $\check{c}_j(a, \zeta) = \alpha c_j(a, \zeta)$, where $\alpha = \psi^E \left(\frac{\beta}{\beta^E} \right)^{\frac{1}{\gamma}}$. Additionally, the $\hat{\hat{u}}$ agent sets $\check{c}_j(a, \zeta)$ such that $x\check{c}_j(a, \zeta)^{-\gamma} = \frac{\partial v_j^R(a,\zeta)}{\partial a}$. Using these properties in equation (48) gives:

$$\frac{c_j(a,\zeta)^{1-\gamma}}{1-\gamma} - x\alpha^{-\gamma} c_j(a,\zeta)^{1-\gamma} = \frac{x(\alpha c_j(a,\zeta))^{1-\gamma}}{1-\gamma} - x(\alpha c_j(a,\zeta))^{1-\gamma}.$$

This can be rearranged to yield:

$$x = \frac{\alpha^\gamma}{1 - \gamma + \alpha\gamma}.$$

Note that $x = \frac{\psi^\gamma}{\beta}$ in the case of sophistication ($\beta^E = \beta$), in which case $\hat{\hat{u}}(c) = \hat{u}(c)$.

Step 2: The Effect of a Consumption Tax. I now introduce a constant perpetual consumption tax of $\tau \in [0, 1)$. Given consumption tax $\tau \in [0, 1)$, let $\check{c}_j(a, \zeta)$ denote the gross consumption expenditure rate of the standard exponential agent.[55] Here I show that a consumption tax of τ does not affect the exponential agent's gross consumption expenditure.

With no tax, the standard exponential agent chooses consumption to maximize $\check{v}$:

$$\check{v}_j(a, \zeta) = \max_{\check{c}} \ \mathbb{E}_t \left[\int_t^\infty e^{-\rho(s-t)} u(\check{c}_s) ds \right].$$

With a consumption tax, the standard exponential agent chooses consumption to maximize:

$$\check{v}_j(a, \zeta; \tau) = \max_{\check{c}} \ \mathbb{E}_t \left[\int_t^\infty e^{-\rho(s-t)} u((1-\tau)\check{c}_s) ds \right].$$

Note that $u((1-\tau)c)$ is a positive affine transformation of $u(c)$. Thus, policy function $\check{c}_j(a, \zeta)$ is unaffected by consumption tax τ. The only effect of the tax is that for $\tau > 0$, $\check{c}_j(a, \zeta)$ denotes gross consumption expenditure. The agent only gets to consume $(1-\tau)\check{c}_j(a, \zeta)$, with the rest going to taxes.

Step 3: The Welfare Effect of Present Bias ($\gamma \neq 1$). Since $\hat{\hat{u}}$ is a positive affine transformation of u, the $\hat{\hat{u}}$ agent behaves identically to a standard exponential agent. Additionally, value function equivalence implies that the realized value function of the IG agent equals the value function of the $\hat{\hat{u}}$ agent whenever $\underline{a}$ does not bind in equilibrium: $v_j^R(a, \zeta) = \hat{\hat{v}}_j(a, \zeta)$. This was shown in Step 1 of this proof.

The final step is to derive the consumption tax τ that equates the realized value function of the IG agent ($v_j^R(a, \zeta)$) with the value function of a standard exponential agent facing a

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[55]In other words, the agent spends $\check{c}$ to consume $(1 - \tau)\check{c}$.

consumption tax $(\check{v}_j(a, \zeta; \tau))$. Using value function equivalence, the realized value function of the IG agent is:

$$v_j^R(a, \zeta) = \hat{v}_j(a, \zeta) = \mathbb{E}_t \left[\int_t^\infty e^{-\rho(s-t)} \hat{u}(\hat{c}_s) ds \right].$$
(49)

The value function of a standard exponential agent facing a consumption tax is:

$$\check{v}_j(a, \zeta; \tau) = \mathbb{E}_t \left[\int_t^\infty e^{-\rho(s-t)} u((1-\tau)\check{c}_s) ds \right].$$
(50)

The key to this proof is to note that $\hat{c}_j(a, \zeta) = \check{c}_j(a, \zeta)$. Therefore the consumption path in the integral of equation (49) is identical to the gross consumption expenditure path in equation (50) (this hold state by state, so it also holds in expectation). Thus, setting equation (49) equal to equation (50) is as simple as finding the value of τ such that:

$$\hat{u}(c) = u((1-\tau)c).$$

This implies that $x = (1-\tau)^{1-\gamma}$. Rearranging gives

$$\tau = 1 - \left(\frac{\alpha^\gamma}{1 - \gamma + \gamma\alpha} \right)^{\frac{1}{1-\gamma}}.$$

Special Case: $\gamma = 1$. In the special case of $\gamma = 1$ the naif and the sophisticate behave identically (Proposition 16). The realized value function $v_j^R(a, \zeta)$ is therefore independent of β^E. So, I calculate the $\gamma = 1$ case under the assumption of sophistication, $\beta^E = \beta$.

I again derive the consumption tax τ that equates the realized value function of the IG agent $(v_j^R(a, \zeta))$ with the value function of a standard exponential agent facing a consumption tax $(\check{v}_j(a, \zeta; \tau))$. Using value function equivalence, the realized value function of the IG agent is:

$$v_j^R(a, \zeta) = \hat{v}_j(a, \zeta) = \mathbb{E}_t \left[\int_t^\infty e^{-\rho(s-t)} \hat{u}(\hat{c}_s) ds \right].$$
(51)

The value function of a standard exponential agent facing a consumption tax is:

$$\check{v}_j(a, \zeta; \tau) = \mathbb{E}_t \left[\int_t^\infty e^{-\rho(s-t)} u((1-\tau)\check{c}_s) ds \right]. \tag{52}$$

Since $\hat{c}_j(a, \zeta) = \check{c}_j(a, \zeta)$, the consumption path in the integral of equation (51) is identical to the gross consumption expenditure path in equation (52). As above, I need to find the value of τ such that:

$$\hat{u}(c) = u((1-\tau)c).$$

When $\gamma = 1$ this implies that $-\ln(\beta) + \frac{\beta-1}{\beta} = \ln(1-\tau)$. Rearranging gives

$$\tau = 1 - \frac{\exp\left(\frac{\beta-1}{\beta}\right)}{\beta}.$$

The Effect of β and β^E. First, I show that τ is decreasing in α. The derivative

$$\frac{\partial \tau}{\partial \alpha} = \frac{-1}{1-\gamma} \left(\frac{\alpha^\gamma}{1-\gamma+\gamma\alpha} \right)^{\frac{\gamma}{1-\gamma}} \left(\frac{\gamma\alpha^{\gamma-1}}{1-\gamma+\gamma\alpha} - \frac{\gamma\alpha^\gamma}{(1-\gamma+\gamma\alpha)^2} \right)$$

implies that

$$\text{sgn}\left(\frac{\partial \tau}{\partial \alpha} \right) = \text{sgn}(\gamma - 1) \times \text{sgn}\left(1 - \frac{\alpha}{1-\gamma+\gamma\alpha} \right), \text{ or equivalently}$$

$$\text{sgn}\left(\frac{\partial \tau}{\partial \alpha} \right) = \text{sgn}(\gamma - 1)\text{sgn}(1-\gamma).$$

Thus, τ is always decreasing in α.

The derivative of α with respect to β is:

$$\frac{\partial \alpha}{\partial \beta} = \frac{\psi^E}{\gamma \beta^E} \left(\frac{\beta}{\beta^E} \right)^{\frac{1-\gamma}{\gamma}} > 0.$$

As stated in the main text, this implies that $\frac{\partial \tau}{\partial \beta} < 0$.

The derivative of α with respect to β^E is:

$$\frac{\partial \alpha}{\partial \beta^E} = \frac{1}{\gamma} \left(\frac{\beta}{\beta^E} \right)^{\frac{1}{\gamma}} - \frac{1}{\gamma} \left(\frac{\beta}{\beta^E} \right)^{\frac{1}{\gamma}} \frac{\psi^E}{\beta^E}.$$

So, $\frac{\partial \alpha}{\partial \beta} > 0$ when $\beta^E > \psi^E$, and $\frac{\partial \alpha}{\partial \beta} < 0$ when $\beta^E < \psi^E$. Since $\beta^E > \psi^E$ when $\gamma < 1$ (and vice versa), this implies that α is increasing in β^E when $\gamma < 1$, and decreasing in β^E when $\gamma > 1$. This also implies that $\frac{\partial \tau}{\partial \beta^E} < 0$ when $\gamma < 1$, and $\frac{\partial \tau}{\partial \beta^E} > 0$ when $\gamma > 1$. As stated in the main text, naivete increases the welfare cost of present bias when $\gamma > 1$.

C.13   Proof of Proposition 21

This follows from the proof of Proposition 20, which shows that the realized continuation-value function of the IG agent is a positive affine transformation of the value function for the standard exponential agent. Accordingly, improving the (realized) welfare of the IG agent is equivalent to improving the welfare of the standard exponential agent.

D   Model Solution with Naivete

This section replicates the numerical example of Section 3.3 under the assumption of complete naivete. To generate an equilibrium interest rate of 3%, I set $\beta = 0.75$, $\beta^E = 1$, and $\rho = 2.45\%$. The calibration is otherwise identical to Section 3.3.

Overall the results are qualitatively similar. The biggest difference between the consumption of the naif and the sophisticate occurs near $\underline{a}$. Though the naif still overconsumes near the borrowing constraint, Figure 5 illustrates that the naif overconsumes by less than the sophisticate. As described in the main text, when the consumer is sophisticated their present bias interacts with the effective planning horizon to increase consumption near $\underline{a}$. This effect does not arise under naivete because the naif trusts all future selves.
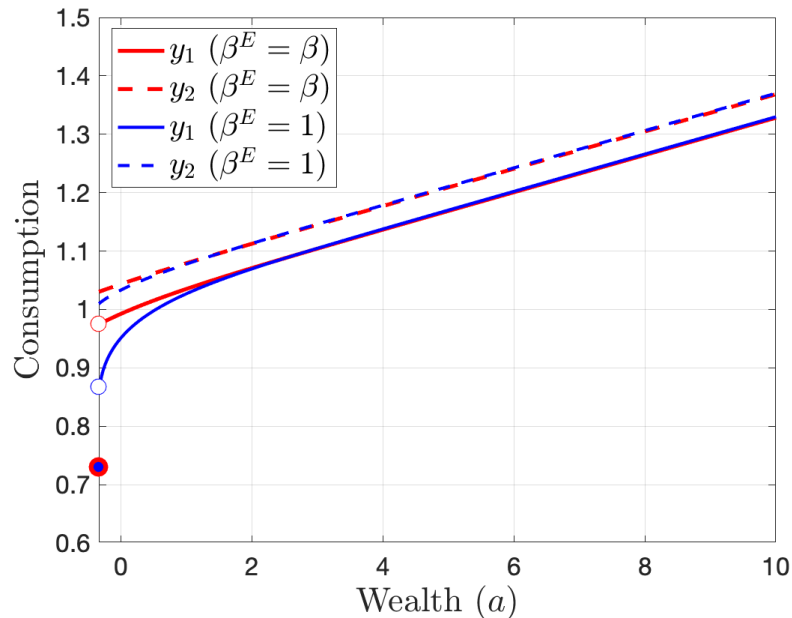

Figure 5: **Equilibrium Consumption-Saving Decisions.** The figure plots the equilibrium consumption function for the $\beta^E = \beta$ calibration (sophistication) and the $\beta^E = 1$ calibration (naivete).
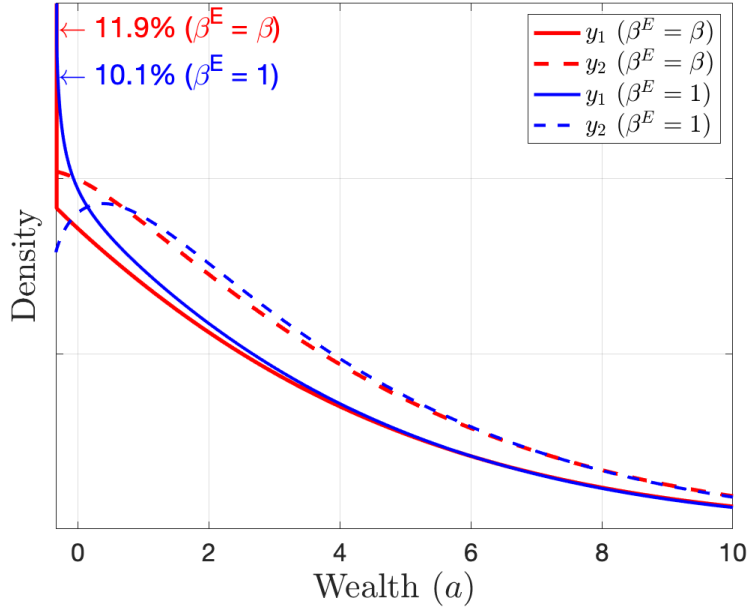
Figure 6: **The Distribution of Wealth.** This figure shows the stationary wealth distribution for the $\beta^E = \beta$ calibration and the $\beta^E = 1$ calibration.
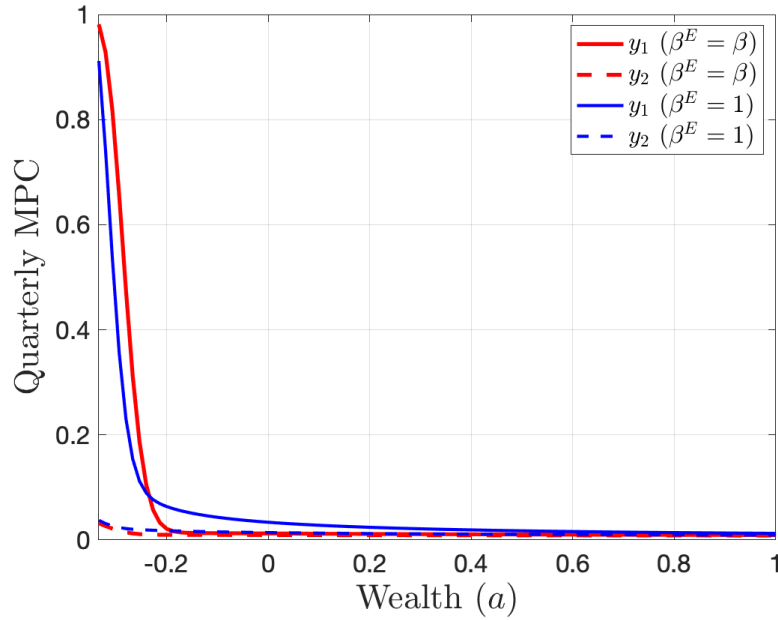

Figure 7: **MPCs.** This figure plots quarterly MPCs for the $\beta^E = \beta$ calibration and the $\beta^E = 1$ calibration.